The quadratic linking degree

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Knot theory in a nutshell 1

Topological objects of interest are knots and links.

- A **knot** is a (closed) topological subspace of the 3-sphere $\mathbb{S}^3$ which is homeomorphic to the circle $\mathbb{S}^1$.
- An **oriented knot** is a knot with a "continuous" local trivialization of its tangent bundle, or equivalently of its normal bundle (the ambient space being oriented). There are two orientation classes.

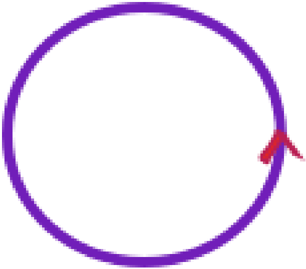


Figure: The unknot

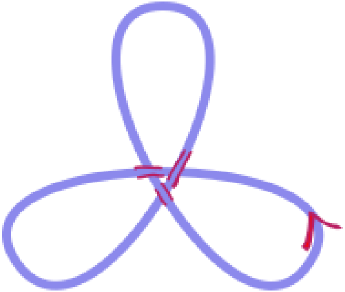


Figure: The trefoil knot

Knot theory in a nutshell 2

- A **link** is a finite union of disjoint knots. A link is **oriented** if all its components (i.e. its knots) are oriented.
- The **linking number** of an (oriented) link with two components is the number of times one of the components turns around the other component.

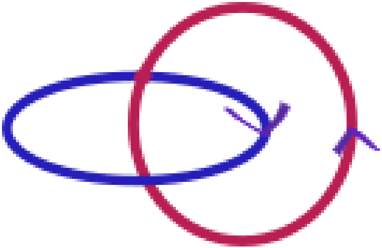


Figure: The Hopf link

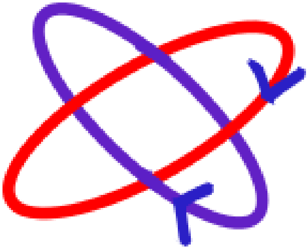
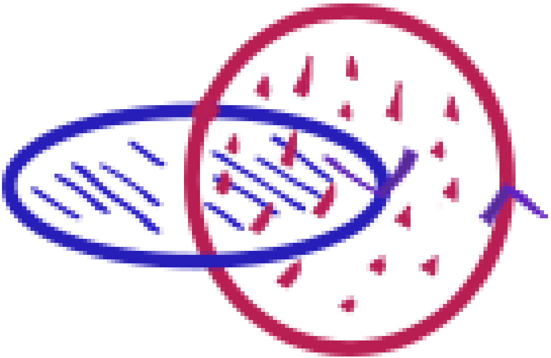
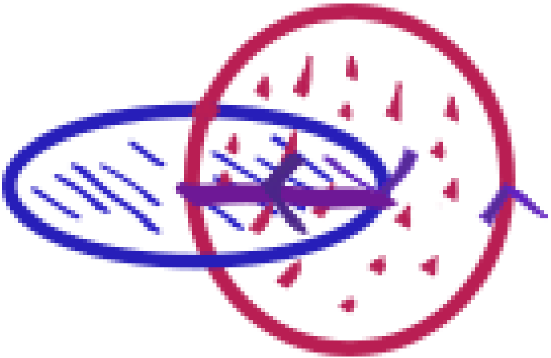


Figure: The Solomon link

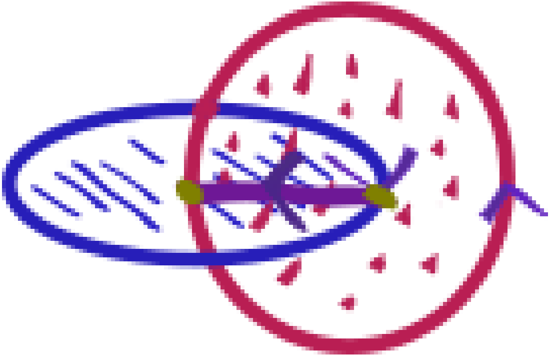
Defining the linking number: Seifert surfaces



Defining the linking number: intersection of S. surfaces



Defining the linking number: boundary of int. of S. surf.



The formal definition of the linking number

Let $L = K_1 \sqcup K_2$ be an oriented link with two components.

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Oriented fundamental class and Seifert class

Let $i \in \{1, 2\}$. The class S_i in $H^1(\mathbb{S}^3 \setminus L) \simeq H_2^{\mathrm{BM}}(\mathbb{S}^3, L)$ of Seifert surfaces of the oriented knot K_i is the unique class that is sent by the boundary map to the (oriented) fundamental class of K_i in $H^0(K_i) \subset H^0(L)$.

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Linking class and linking number

The linking class of L is the image of the cup-product $S_1 \cup S_2 \in H^2(\mathbb{S}^3 \setminus L)$ by the boundary map $\partial : H^2(\mathbb{S}^3 \setminus L) \to H^1(L)$. The linking number of $L = K_1 \sqcup K_2$ is the integer $n \in \mathbb{Z}$ such that the linking class in $H^1(L) = \mathbb{Z}[\omega_{K_1}] \oplus \mathbb{Z}[\omega_{K_2}]$ is equal to $(n[\omega_{K_1}], -n[\omega_{K_2}])$ (where ω_{K_i} is the volume form of the oriented knot K_i).

Homotopies in a nutshell

Homotopic maps

Two continuous maps $f, g : X \to Y$ are homotopic if there exists a homotopy from f to g, i.e. a continuous map $H : X \times [0,1] \to Y$ such that for all $x \in X$, $H(x,0) = f(x)$ and $H(x,1) = g(x)$.

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Homotopy types of topological spaces

Two topological spaces X and Y have the same homotopy type if there exists a homotopy equivalence from X to Y, i.e. a couple $(i : X \to Y, j : Y \to X)$ of continuous maps such that $j \circ i$ is homotopic to the identity of X and $i \circ j$ is homotopic to the identity of Y.

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Important example

For all $n \geq 1$, $\mathbb{S}^n$ has the same homotopy type as $\mathbb{R}^{n+1} \setminus \{0\}$.

Oriented links in algebraic geometry 1

Link with two components

A link with two components is a couple of closed immersions $\varphi_i : \mathbb{A}^2_F \setminus \{0\} \to \mathbb{A}^4_F \setminus \{0\}$ with disjoint images Z_i (where $i \in \{1, 2\}$).

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An orientation o_i of Z_i is an isomorphism from the determinant (i.e. the maximal exterior power) of the normal sheaf $\mathcal{N}_{Z_i/\mathbb{A}^4_F \setminus \{0\}}$ of Z_i in $\mathbb{A}^4_F \setminus \{0\}$ to the tensor product of an invertible $\mathcal{O}_{Z_i}$-module $\mathcal{L}_i$ with itself:

$$o_i : \nu_{Z_i} := \det(\mathcal{N}_{Z_i/\mathbb{A}^4_F \setminus \{0\}}) \simeq \mathcal{L}_i \otimes \mathcal{L}_i$$

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Orientation classes

Two orientations $o_i : \nu_{Z_i} \to \mathcal{L}_i \otimes \mathcal{L}_i$ and $o'_i : \nu_{Z_i} \to \mathcal{L}'_i \otimes \mathcal{L}'_i$ of Z_i represent the same orientation class of Z_i if there exists an isomorphism $\psi : \mathcal{L}_i \simeq \mathcal{L}'_i$ such that $(\psi \otimes \psi) \circ o_i = o'_i$.

Oriented links in algebraic geometry 2

Oriented link with two components

An oriented link with two components is a link with two components $(\varphi_1 : \mathbb{A}^2_F \setminus \{0\} \to Z_1, \varphi_2 : \mathbb{A}^2_F \setminus \{0\} \to Z_2)$ together with an orientation class $\overline{o_1}$ of Z_1 and an orientation class $\overline{o_2}$ of Z_2.

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The orientation classes of Z_i are parametrized by the elements of $F^*/(F^*)^2$ (where $(F^*)^2 = \{a \in F^*, \exists b \in F^*, a = b^2\}$).

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If $F = \mathbb{Q}$ then $F^*/(F^*)^2$ has infinitely many elements (the classes of the integers without square factors).

The Hopf link

We fix coordinates x, y, z, t for $\mathbb{A}^4_F$ and u, v for $\mathbb{A}^2_F$ once and for all.

- The image of the Hopf link:

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- The orientation of the Hopf link:

$$o_1 : \overline{x}^* \wedge \overline{y}^* \mapsto 1 \otimes 1, o_2 : \overline{z}^* \wedge \overline{t}^* \mapsto 1 \otimes 1$$

A variant of the Hopf link

- The image is the same as the Hopf link's image:

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Chow groups and intersection theory

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- You may know the following exact sequence where $Y \subset X$ is closed:

$$CH_p(Y) \longrightarrow CH_p(X) \longrightarrow CH_p(X \setminus Y) \longrightarrow 0$$

It can be extended into the following long exact sequence:

$$\cdots \to A_{p+1}(X \setminus Y, -p) \to CH_p(Y) \to CH_p(X) \to CH_p(X \setminus Y) \to 0$$

Chow-Witt groups and quadratic intersection theory

- Solution to the second problem (orientations): replace (generalised) Chow groups, a.k.a. Rost groups, with (generalised) Chow-Witt groups, a.k.a. Rost-Schmid groups; see for instance the chapter *Lectures on Chow-Witt groups* by Jean Fasel in the book *Motivic homotopy theory and refined enumerative geometry* (2020)

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- To a smooth F-scheme Y, an integer $j \in \mathbb{Z}$ and an invertible $\mathcal{O}_Y$-module $\mathcal{L}$ we associate the corresponding Rost-Schmid complex $\bigoplus_{i \geq 0} \bigoplus_{p \text{ point of codim } i \text{ in } Y} K^{\mathrm{MW}}_{j-i}(\kappa(p)) \otimes_{\mathbb{Z}[\kappa(p)^*]} \mathbb{Z}[(\nu_p \otimes \mathcal{L}_{|p}) \setminus \{0\}]$.

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- The i-th cohomology group, called Rost-Schmid group, is denoted $H^i(Y, \underline{K}^{\mathrm{MW}}_j\{\mathcal{L}\})$. If $j = i$ then we call $H^i(Y, \underline{K}^{\mathrm{MW}}_i\{\mathcal{L}\})$ the i-th Chow-Witt group of Y twisted by $\mathcal{L}$ and denote it $\widetilde{CH}^i(Y, \mathcal{L})$. We have a canonical morphism $\widetilde{CH}^i(Y, \mathcal{L}) \to CH^i(Y)$.

Intersection product

We have an intersection product

$$\cdot : H^i(Y, \underline{K}_j^{\mathrm{MW}}) \times H^{i'}(Y, \underline{K}_{j'}^{\mathrm{MW}}) \to H^{i+i'}(Y, \underline{K}_{j+j'}^{\mathrm{MW}})$$

which makes $\bigoplus_{i \in \mathbb{N}_0, j \in \mathbb{Z}} H^i(Y, \underline{K}_j^{\mathrm{MW}})$ into a graded $K_0^{\mathrm{MW}}(F)$-algebra.

In particular, we have $\cdot : \widetilde{CH}^i(Y) \times \widetilde{CH}^{i'}(Y) \to \widetilde{CH}^{i+i'}(Y)$ which makes $\bigoplus_{i \in \mathbb{N}_0} \widetilde{CH}^i(Y)$ into a graded $K_0^{\mathrm{MW}}(F)$-algebra (the Chow-Witt ring).

Boundary maps and the localization long exact sequence

If $i : Z \to X$ is a closed subscheme and $j : U \to X$ is the complementary open subscheme, Z, U, X being smooth F-schemes (with F a perfect field) of pure dimensions d_Z, d and d, then for each n, m there is a boundary map $\partial : H^{n+d_X-d_Z}(U, \underline{K}^{\mathrm{MW}}_{m+d_X-d_Z}) \to H^{n+1}(Z, \underline{K}^{\mathrm{MW}}_m\{\nu_Z\})$ such that the following is a long exact sequence:

$$\ldots \longrightarrow H^n(Z, \underline{K}^{\mathrm{MW}}_m\{\nu_Z\}) \xrightarrow{i_*} H^{n+d_X-d_Z}(X, \underline{K}^{\mathrm{MW}}_{m+d_X-d_Z}) \xrightarrow{j^*}$$

$$\xrightarrow{j^*} H^{n+d_X-d_Z}(U, \underline{K}^{\mathrm{MW}}_{m+d_X-d_Z}) \xrightarrow{\partial} H^{n+1}(Z, \underline{K}^{\mathrm{MW}}_m\{\nu_Z\}) \longrightarrow \ldots$$

Milnor-Witt K-theory and quadratic forms

The generators of the Milnor-Witt K-theory ring of a field F are denoted $[a] \in K_1^{\mathrm{MW}}(F)$ for every $a \in F^*$ and $\eta \in K_{-1}^{\mathrm{MW}}(F)$. We denote $\langle a \rangle = \eta[a] + 1 \in K_0^{\mathrm{MW}}(F)$ for every $a \in F^*$.

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The (commutative) ring with unit $K_0^{\mathrm{MW}}(F)$ is isomorphic to the Grothendieck-Witt ring $\mathrm{GW}(F)$ of F via $\langle a \rangle \in K_0^{\mathrm{MW}}(F) \leftrightarrow \langle a \rangle \in \mathrm{GW}(F)$. For all $n < 0$, the abelian group $K_n^{\mathrm{MW}}(F)$ is isomorphic to the Witt group $\mathrm{W}(F)$ of F via $\langle a \rangle \eta^{-n} \in K_n^{\mathrm{MW}}(F) \leftrightarrow \langle a \rangle \in \mathrm{W}(F)$.

For all $a \in F^*$, $\langle a \rangle$ is the equivalence class of the symmetric bilinear form $\begin{cases} F \times F & \to & F \\ (x,y) & \mapsto & axy \end{cases}$ or, if F is of characteristic $\neq 2$, of the quadratic form $\begin{cases} F & \to & F \\ x & \mapsto & ax^2 \end{cases}$. $\mathrm{GW}(F)$ is made up of $\mathbb{Z}$-linear combinations of $\langle a \rangle$ and $\mathrm{W}(F) = \mathrm{GW}(F)/(\langle 1 \rangle + \langle -1 \rangle)$ is made up of sums of $\langle a \rangle$.

Let $n \geq 2$ be an integer, $i \in \mathbb{N}_0, j \in \mathbb{Z}$. The Rost-Schmid group

$$H^i(\mathbb{A}^n_F \setminus \{0\}, \underline{K}^{\mathrm{MW}}_j) \text{ is isomorphic to } \begin{cases} K^{\mathrm{MW}}_j(F) & \text{if } i = 0 \\ K^{\mathrm{MW}}_{j-n}(F) & \text{if } i = n-1 \\ 0 & \text{otherwise} \end{cases}.$$

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This is similar to the fact in classical homotopy theory that $H^i(\mathbb{S}^{n-1})$ is isomorphic to $\begin{cases} \mathbb{Z} & \text{if } i = 0 \\ \mathbb{Z} & \text{if } i = n-1 \\ 0 & \text{otherwise} \end{cases}$.

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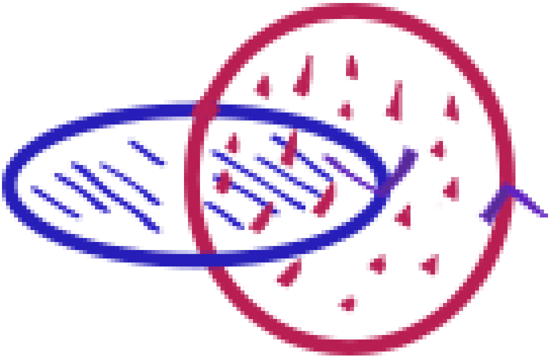
In particular, $H^1(\mathbb{A}^2_F \setminus \{0\}, \underline{K}^{\mathrm{MW}}_0) \simeq K^{\mathrm{MW}}_{-2}(F)$. We can fix such an isomorphism, but it is not canonical.

The linking number and the quadratic linking degree

Let $L = K_1 \sqcup K_2$ be an oriented link (in knot theory) and $\mathscr{L}$ be an oriented link with two components (in motivic knot theory), i.e. a couple of closed immersions $\varphi_i : \mathbb{A}^2_F \setminus \{0\} \to \mathbb{A}^4_F \setminus \{0\}$ with disjoint images Z_i and orientation classes $\overline{o_i}$. We denote $Z := Z_1 \sqcup Z_2$.

Step 1 in a picture: Seifert surfaces



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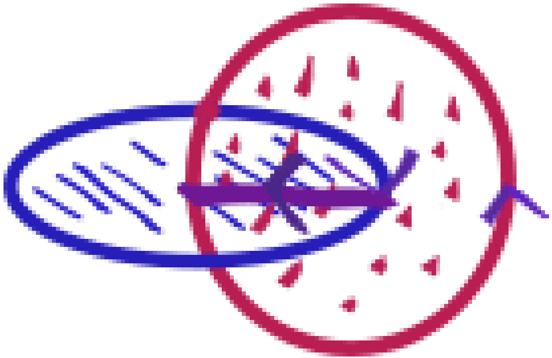
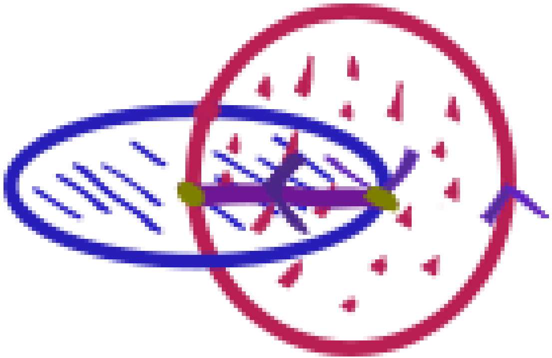
Motivic knot theory

We define an analogue $[o_i] \in H^0(Z_i, \underline{K}_{-1}^{\mathrm{MW}}\{\nu_{Z_i}\})$ of the oriented fundamental class of each oriented component of $\mathscr{L}$ then we define the Seifert class $\mathcal{S}_i$ as the unique class in $H^1(X \setminus Z, \underline{K}_1^{\mathrm{MW}})$ that is sent by the boundary map to the oriented fundamental class $[o_i] \in H^0(Z, \underline{K}_{-1}^{\mathrm{MW}}\{\nu_Z\})$.

Step 2 in two pictures: intersection of Seifert surfaces



Step 2 in two pictures: boundary of int. of S. surfaces



Step 2

Knot theory

The linking class of L is the image of the cup-product $S_1 \cup S_2 \in H^2(\mathbb{S}^3 \setminus L)$ by the boundary map $\partial : H^2(\mathbb{S}^3 \setminus L) \to H^1(L)$.

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Motivic knot theory

We define the quadratic linking class of $\mathscr{L}$ as the image of the intersection product $\mathcal{S}_1 \cdot \mathcal{S}_2 \in H^2(X \setminus Z, \underline{K}_2^{\mathrm{MW}})$ by the boundary map $\partial : H^2(X \setminus Z, \underline{K}_2^{\mathrm{MW}}) \to H^1(Z, \underline{K}_0^{\mathrm{MW}}\{\nu_Z\})$.

Step 3

Knot theory

The linking number of $L = K_1 \sqcup K_2$ is the integer $n \in \mathbb{Z}$ such that the linking class in $H^1(L) = \mathbb{Z}[\omega_{K_1}] \oplus \mathbb{Z}[\omega_{K_2}]$ is equal to $(n[\omega_{K_1}], -n[\omega_{K_2}])$ (where ω_{K_i} is the volume form of the oriented knot K_i).

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Motivic knot theory

We define the quadratic linking degree of $\mathscr{L}$ as the image of the quadratic linking class of $\mathscr{L}$ by the isomorphism
$H^1(Z, \underline{K}_0^{\mathrm{MW}}\{\nu_Z\}) \to H^1(Z, \underline{K}_0^{\mathrm{MW}}) \to$
$H^1(\mathbb{A}^2_F \setminus \{0\}, \underline{K}_0^{\mathrm{MW}}) \oplus H^1(\mathbb{A}^2_F \setminus \{0\}, \underline{K}_0^{\mathrm{MW}}) \to \mathrm{W}(F) \oplus \mathrm{W}(F)$.

We fixed an isomorphism $H^1(\mathbb{A}^2_F \setminus \{0\}, \underline{K}_0^{\mathrm{MW}}) \to K_{-2}^{\mathrm{MW}}(F)$ once and for all and there is a canonical isomorphism $K_{-2}^{\mathrm{MW}}(F) \to \mathrm{W}(F)$.

The Hopf link

Recall that we fixed coordinates x, y, z, t for $\mathbb{A}^4_F$ and u, v for $\mathbb{A}^2_F$.

- The image of the Hopf link:

$$\{x = 0, y = 0\} \sqcup \{z = 0, t = 0\} \subset \mathbb{A}^4_F \setminus \{0\}$$

- The parametrization of the Hopf link:

$$\varphi_1 : (x, y, z, t) \leftrightarrow (0, 0, u, v), \varphi_2 : (x, y, z, t) \leftrightarrow (u, v, 0, 0)$$

- The orientation of the Hopf link:

$$o_1 : \overline{x}^* \wedge \overline{y}^* \mapsto 1, o_2 : \overline{z}^* \wedge \overline{t}^* \mapsto 1$$

The quadratic linking degree of the Hopf link

Or. fund. classes	$\eta \otimes (\overline{x}^* \wedge \overline{y}^*)$	|	$\eta \otimes (\overline{z}^* \wedge \overline{t}^*)$
Seifert classes	$\langle x \rangle \otimes \overline{y}^*$	|	$\langle z \rangle \otimes \overline{t}^*$
Apply int. prod.	$\langle xz \rangle \otimes (\overline{t}^* \wedge \overline{y}^*)$		
Quad. link. class	$-\langle z \rangle \eta \otimes (\overline{t}^* \wedge \overline{x}^* \wedge \overline{y}^*)$	$\oplus$	$\langle x \rangle \eta \otimes (\overline{y}^* \wedge \overline{z}^* \wedge \overline{t}^*)$
Apply $\widetilde{o}_1 \oplus \widetilde{o}_2$	$-\langle z \rangle \eta \otimes \overline{t}^*$	$\oplus$	$\langle x \rangle \eta \otimes \overline{y}^*$
Apply $\varphi_1^* \oplus \varphi_2^*$	$-\langle u \rangle \eta \otimes \overline{v}^*$	$\oplus$	$\langle u \rangle \eta \otimes \overline{v}^*$
Apply $\partial \oplus \partial$	$-\eta^2 \otimes (\overline{u}^* \wedge \overline{v}^*)$	$\oplus$	$\eta^2 \otimes (\overline{u}^* \wedge \overline{v}^*)$
Quad. link. degree	-1	$\oplus$	1

A variant of the Hopf link

- The image is the same as the Hopf link's image:

$$\{x = y, y = 0\} \sqcup \{z = 0, a \times t = 0\} \subset \mathbb{A}^4_F \setminus \{0\} \text{ with } a \in F^*$$

- The parametrization is the same:

$$\varphi_1 : (x, y, z, t) \leftrightarrow (0, 0, u, v), \varphi_2 : (x, y, z, t) \leftrightarrow (u, v, 0, 0)$$

- The orientation is different:

$$o_1 : \overline{x - y}^* \wedge \overline{y}^* \mapsto 1, o_2 : \overline{z}^* \wedge \overline{at}^* \mapsto 1$$

The quadratic linking degree of a variant of the Hopf link

$[o_1^{var}] = \eta \otimes \overline{x-y}^* \wedge \overline{y}^* = [o_1^{Hopf}]$ $[o_2^{var}] = \eta \otimes \overline{z}^* \wedge \overline{at}^* = \langle a \rangle [o_2^{Hopf}]$

since $\begin{pmatrix} x-y \\ y \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$ since $\begin{pmatrix} z \\ at \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix} \begin{pmatrix} z \\ t \end{pmatrix}$

$\mathcal{S}_1^{var} = \mathcal{S}_1^{Hopf}$ $\mathcal{S}_2^{var} = \langle a \rangle \mathcal{S}_2^{Hopf}$

$$\mathcal{S}_1^{var} \cdot \mathcal{S}_2^{var} = \langle a \rangle \mathcal{S}_1^{Hopf} \cdot \mathcal{S}_2^{Hopf}$$
$$\partial(\mathcal{S}_1^{var} \cdot \mathcal{S}_2^{var}) = \langle a \rangle \partial(\mathcal{S}_1^{Hopf} \cdot \mathcal{S}_2^{Hopf})$$

The quadratic linking degree is $(-\langle a \rangle, 1)$.

Fact

Let $\mathscr{L}$ be an oriented link with two components of quadratic linking degree $(d_1, d_2) \in \mathrm{W}(F) \oplus \mathrm{W}(F)$. Let $a = (a_1, a_2)$ be a couple of elements of F^* and $\mathscr{L}_a$ be the link obtained from $\mathscr{L}$ by changing the orientation o_1 into $o_1 \circ (\times a_1)$ and the orientation o_2 into $o_2 \circ (\times a_2)$. Then $\mathrm{Qlc}_{\mathscr{L}_a} = \langle a_1 a_2 \rangle \, \mathrm{Qlc}_{\mathscr{L}}$ and $\mathrm{Qld}_{\mathscr{L}_a} = (\langle a_2 \rangle d_1, \langle a_1 \rangle d_2)$.

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Similarly, changes of parametrizations of the link can only multiply each component of the quadratic linking degree by elements of the form $\langle a \rangle$ with $a \in F^*$ (and do not change the quadratic linking class).

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We want invariants of the quadratic linking degree. (Similarly to the absolute value of the linking number in knot theory.)

Invariants by multiplication by $\langle a \rangle$ for all $a \in F^*$

Case $F = \mathbb{R}$

If $F = \mathbb{R}$, the absolute value of an element of $\mathrm{W}(\mathbb{R}) \simeq \mathbb{Z}$ is invariant by multiplication by $\langle a \rangle$ for all $a \in F^*$.

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General case

The rank modulo 2 is invariant by multiplication by $\langle a \rangle$ for all $a \in F^*$.

Definition

Let $d \in \mathrm{W}(F)$. There exists a unique sequence of abelian groups $Q_{d,k}$ and of elements $\Sigma_k(d) \in Q_{d,k}$, where k ranges over the nonnegative even integers, such that:

- $Q_{d,0} = \mathrm{W}(F)$ and $\Sigma_0(d) = 1 \in Q_{d,0}$;
- for each positive even integer k, $Q_{d,k}$ is the quotient group $Q_{d,k-2}/(\Sigma_{k-2}(d))$;
- for each positive even integer k, $\Sigma_k(d) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \langle \prod_{1 \leq j \leq k} a_{i_j} \rangle \in Q_{d,k}$ whenever $d = \sum_{i=1}^{n} \langle a_i \rangle \in \mathrm{W}(F)$.

General case

The Σ_k are invariant by multiplication by $\langle a \rangle$ for all $a \in F^*$.

- $\Sigma_2 : \begin{cases} W(F) & \rightarrow & W(F)/(1) \\ \sum_{i=1}^{n} \langle a_i \rangle & \mapsto & \sum_{1 \leq i < j \leq n} \langle a_i a_j \rangle \end{cases}$ (if $n < 2$, it sends $\sum_{i=1}^{n} \langle a_i \rangle$ to 0)

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- $\Sigma_4 : \begin{cases} \mathrm{W}(F) & \to & \bigcup_{d \in \mathrm{W}(F)} (\mathrm{W}(F)/(1))/(\Sigma_2(d)) \\ \sum_{i=1}^{n} \langle a_i \rangle & \mapsto & \sum_{1 \leq i < j < k < l \leq n} \langle a_i a_j a_k a_l \rangle \end{cases}$

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- We only want to compare $\Sigma_4(d)$ and $\Sigma_4(d')$ if $\Sigma_2(d) = \Sigma_2(d')$.

Another Hopf link

From now on, F is a perfect field of characteristic different from 2. Recall that we fixed coordinates x, y, z, t for $\mathbb{A}^4_F$ and u, v for $\mathbb{A}^2_F$.

- The image is different from the Hopf link we saw before:

$$\{z = x, t = y\} \sqcup \{z = -x, t = -y\} \subset \mathbb{A}^4_F \setminus \{0\}$$

  But the change of coordinates $x' = z - x$, $y' = t - y$, $z' = z + x$, $t' = t + y$ would give $\{x' = 0, y' = 0\} \sqcup \{z' = 0, t' = 0\} \subset \mathbb{A}^4_F \setminus \{0\}$.

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- The orientation is the following:

$$o_1 : \overline{z - x}^* \wedge \overline{t - y}^* \mapsto 1, o_2 : \overline{z + x}^* \wedge \overline{t + y}^* \mapsto 1$$

- This Hopf link is an analogue of the Hopf link in knot theory! In knot theory, the Hopf link is given by $\{z = x, t = y\} \sqcup \{z = -x, t = -y\}$ in $\mathbb{S}^3_\varepsilon = \{(x, y, z, t) \in \mathbb{R}^4, x^2 + y^2 + z^2 + t^2 = \varepsilon^2\}$ for ε small enough and has linking number 1.

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- If $F = \mathbb{R}$, the absolute value of each component is 1.
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- If $F = \mathbb{R}$, the absolute value of each component is 1.
- The rank modulo 2 of each component is 1.
- For every positive even integer k, the image by Σ_k of each component is 0.

The Solomon link

- In knot theory, the Solomon link is given by $\{z = x^2 - y^2, t = 2xy\} \sqcup \{z = -x^2 + y^2, t = -2xy\}$ in $\mathbb{S}^3_\varepsilon$ for ε small enough and has linking number 2.
- In motivic knot theory, the image of the Solomon link is:

$$\{z = x^2 - y^2, t = 2xy\} \sqcup \{z = -x^2 + y^2, t = -2xy\} \subset \mathbb{A}^4_F \setminus \{0\}$$

- The parametrization is $\varphi_1 : (x, y, z, t) \leftrightarrow (u, v, u^2 - v^2, 2uv)$ and $\varphi_2 : (x, y, z, t) \leftrightarrow (u, v, -u^2 + v^2, -2uv)$.
- The orientation is the following:

$$o_1 : \overline{z - x^2 + y^2}^* \wedge \overline{t - 2xy}^* \mapsto 1, o_2 : \overline{z + x^2 - y^2}^* \wedge \overline{t + 2xy}^* \mapsto 1$$

- Its quadratic linking degree is $(\langle 1\rangle + \langle 1\rangle, \langle -1\rangle + \langle -1\rangle) = (2, -2) \in \mathrm{W}(F) \oplus \mathrm{W}(F)$.

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- If $F = \mathbb{R}$, the absolute value of each component is 2.
- The rank modulo 2 of each component is 0.
- For every positive even integer k, the image by Σ_k of each component is 0.
- More generally, we have analogues of the torus links $T(2, 2n)$ (of linking number n); the quadratic linking degree of $T(2, 2n)$ is $(n, -n) \in \mathrm{W}(F) \oplus \mathrm{W}(F)$, which gives n as absolute value if $F = \mathbb{R}$, n modulo 2 as rank modulo 2, and 0 for the Σ_k.

Binary links

- The image of the binary link B_a with $a \in F^* \setminus \{-1\}$:

$$\{f_1 = 0, g_1 = 0\} \sqcup \{f_2 = 0, g_2 = 0\} \subset \mathbb{A}^4_F \setminus \{0\}$$

with $f_1 = t - ((1+a)x - y)y$, $g_1 = z - x(x-y)$,
$f_2 = t + ((1+a)x - y)y$, $g_2 = z + x(x-y)$.

- The parametrization of the binary link B_a:

$$\varphi_1 : (x, y, z, t) \leftrightarrow (u, v, \quad ((1+a)u - v)v, \quad u(u-v))$$

$$\varphi_2 : (x, y, z, t) \leftrightarrow (u, v, -((1+a)u - v)v, -u(u-v))$$

- The orientation of the binary link B_a:

$$o_1 : \overline{f_1}^* \wedge \overline{g_1}^* \mapsto 1, o_2 : \overline{f_2}^* \wedge \overline{g_2}^* \mapsto 1$$

Or. fund. cyc.	$\eta \otimes (\overline{f_1}^* \wedge \overline{g_1}^*)$	|	$\eta \otimes (\overline{f_2}^* \wedge \overline{g_2}^*)$
Seifert divisors	$\langle f_1 \rangle \otimes \overline{g_1}^*$	|	$\langle f_2 \rangle \otimes \overline{g_2}^*$
Apply inter. prod.	$\langle f_1 f_2 \rangle \otimes (\overline{g_2}^* \wedge \overline{g_1}^*) \cdot (z, x - y)$ $+\langle f_1 f_2 \rangle \otimes (\overline{g_2}^* \wedge \overline{g_1}^*) \cdot (z, x)$		
...	...		
Apply $\partial \oplus \partial$	$(1 + \langle a \rangle)\eta^2 \otimes (\overline{u}^* \wedge \overline{v}^*)$	$\oplus$	$-(1 + \langle a \rangle)\eta^2 \otimes (\overline{u}^* \wedge \overline{v}^*)$
Quad. lk. deg.	$1 + \langle a \rangle$	$\oplus$	$-(1 + \langle a \rangle)$

Or. fund. cyc.	$\eta \otimes (\overline{f_1}^* \wedge \overline{g_1}^*)$		$\eta \otimes (\overline{f_2}^* \wedge \overline{g_2}^*)$
Seifert divisors	$\langle f_1 \rangle \otimes \overline{g_1}^*$		$\langle f_2 \rangle \otimes \overline{g_2}^*$
Apply inter. prod.	$\langle f_1 f_2 \rangle \otimes (\overline{g_2}^* \wedge \overline{g_1}^*) \cdot (z, x - y)$ $+\langle f_1 f_2 \rangle \otimes (\overline{g_2}^* \wedge \overline{g_1}^*) \cdot (z, x)$		
...	...		
Apply $\partial \oplus \partial$	$(1 + \langle a \rangle)\eta^2 \otimes (\overline{u}^* \wedge \overline{v}^*)$	$\oplus$	$-(1 + \langle a \rangle)\eta^2 \otimes (\overline{u}^* \wedge \overline{v}^*)$
Quad. lk. deg.	$1 + \langle a \rangle$	$\oplus$	$-(1 + \langle a \rangle)$

- If we change its orientations and its parametrizations then we get $(\langle a \rangle + \langle b \rangle, \langle ca \rangle + \langle cb \rangle) \in \mathrm{W}(F) \oplus \mathrm{W}(F)$ with $a, b, c \in F^*$ such that $a + b \neq 0$. The rank modulo 2 of each component is 0.

Or. fund. cyc.	$\eta \otimes (\overline{f_1}^* \wedge \overline{g_1}^*)$	|	$\eta \otimes (\overline{f_2}^* \wedge \overline{g_2}^*)$
Seifert divisors	$\langle f_1 \rangle \otimes \overline{g_1}^*$	|	$\langle f_2 \rangle \otimes \overline{g_2}^*$
Apply inter. prod.	$\langle f_1 f_2 \rangle \otimes (\overline{g_2}^* \wedge \overline{g_1}^*) \cdot (z, x-y)$ $+\langle f_1 f_2 \rangle \otimes (\overline{g_2}^* \wedge \overline{g_1}^*) \cdot (z, x)$		
...	...		
Apply $\partial \oplus \partial$	$(1+\langle a \rangle)\eta^2 \otimes (\overline{u}^* \wedge \overline{v}^*)$	$\oplus$	$-(1+\langle a \rangle)\eta^2 \otimes (\overline{u}^* \wedge \overline{v}^*)$
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- If $F = \mathbb{R}$, the absolute value of each component is $\begin{cases} 2 \text{ if } a > 0 \\ 0 \text{ if } a < 0 \end{cases}$.

Or. fund. cyc.	$\eta \otimes (\overline{f_1}^* \wedge \overline{g_1}^*)$	|	$\eta \otimes (\overline{f_2}^* \wedge \overline{g_2}^*)$
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- If $F = \mathbb{R}$, the absolute value of each component is $\begin{cases} 2 \text{ if } a > 0 \\ 0 \text{ if } a < 0 \end{cases}$.
- Σ_2 of each component is $\langle a \rangle \in \mathrm{W}(F)/(1)$. For instance, if $F = \mathbb{Q}$, Σ_2 distinguishes between all the B_p with p prime numbers. $\Sigma_4 = 0$ etc.

Everything new I presented can be found in my preprint "The quadratic linking degree":

- HAL: Clémentine Lemarié--Rieusset. THE QUADRATIC LINKING DEGREE. 2022. ⟨hal-03821736⟩
- arXiv: Clémentine Lemarié--Rieusset. The quadratic linking degree. arXiv:2210.11048 [math.AG]

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Thanks for your attention!