Let $f\colon X \to Y$ be a locally of finite presentation morphism of schemes. The following are equivalent definitions for étale maps (use any in the exercises, except in Exercise 2):

1. f is flat and unramified;
2. f is flat and $\Omega_{X/Y} = 0$;
3. (Infinitesimal Lifting Criterion) for every surjection of rings $A \to A_0$ with nilpotent kernel and every commutative diagram

$$\begin{tikzcd} \operatorname{Spec} A_0 \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \operatorname{Spec} A \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

   of solid arrows, there exists a unique dotted arrow filling in the diagram.

Exercise 1 (Being unramified is a fiberwise condition)**.** Let $f\colon X \to Y$ be a locally finite type morphism. Then f is unramified if and only if for every $y \in Y$, the fiber X_y is a disjoint union of finite separable field extensions of $k(y)$.

Exercise 2. In the characterization of étale maps above, prove that (1) $\implies$ (2).

Exercise 3. Let $f\colon X \to Y$ be an étale morphism. Prove that for any $x \in X$ such that $k(x) = k(f(x))$, the induced map on completions along the maximal ideals

$$\widehat{\mathcal{O}_{Y,f(x)}} \to \widehat{\mathcal{O}_{X,x}}$$

is an isomorphism. Provide an example of an unramified morphism f such that the above does not hold.

Exercise 4 (Homework)**.** Determine whether the following maps are flat, unramified, étale or finite. If it is not étale, find the maximal open of the source where the map is étale (k here is a field):

(a) $\mathbb{A}^1_k \to \mathbb{A}^1_k$, $x \mapsto x^\ell$, for $\ell \geq 1$;

(b) $\operatorname{Spec} k \to \operatorname{Spec} k[\epsilon]/(\epsilon^2)$, given by $\epsilon \mapsto 0$;

(c) $\operatorname{Spec} k[t] \to \operatorname{Spec} k[x,y]/(y^2 - x^3)$, given by $x \mapsto t^2$, $y \mapsto t^3$;

(d) $\bigcup_{\operatorname{ht}(\mathfrak{p})=1} \operatorname{Spec} k[x]_{\mathfrak{p}} \to \operatorname{Spec} k[x]$, induced by localization at primes $\mathfrak{p}$;

(e) $\operatorname{Spec} \mathcal{O}_{\mathbb{Q}(\sqrt{5})} \to \operatorname{Spec} \mathcal{O}_{\mathbb{Q}} = \mathbb{Z}$, where $\mathcal{O}$ denotes the ring of integers.

Exercise 5 (Bonus, Galois coverings)**.** Let G be a finite group acting on a quasi-projective integral curve C over a field k. Show that there exists a quotient $\pi\colon C \to C'$ (i.e. a G-invariant morphism satisfying the universal property of the quotient) and that π is flat. Assuming that $\operatorname{char} k = 0$ and that G acts freely on C, show that π is étale. (Is it possible to generalize this statement in higher dimension?)

Exercise 1 (Frame bundles)**.** Given a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$, we define $\mathbb{A}(\mathcal{F}) \to X$ to be the relative scheme representing $\underline{\mathrm{Spec}}_X(\mathrm{Sym}^* \mathcal{F})$. Verify that this is given by:

$$\begin{aligned} \underline{\mathrm{Spec}}_X(\mathrm{Sym}^* \mathcal{F})\colon \mathrm{Sch}/X &\longrightarrow \mathrm{Sets} \\ (T \xrightarrow{f} X) &\mapsto \Gamma(T, (f^*\mathcal{F})^\vee) \end{aligned}$$

(a) Let X be a noetherian scheme and consider $\mathcal{F}$ as above. Then $\mathbb{A}(\mathcal{F}) \to X$ is smooth if and only if $\mathcal{F}$ is a vector bundle of finite rank.

(b) Let E be a vector bundle over X of rank n. The *frame bundle* Fr_E is the functor $\underline{\mathrm{Isom}}_{\mathcal{O}_X}(\mathcal{O}_X^{\oplus n}, E)$ on Sch/X, i.e.,

$$\begin{aligned} \mathrm{Fr}_E : \mathrm{Sch}/X &\to \mathrm{Sets} \\ (T \to X) &\mapsto \{\text{trivializations } \mathcal{O}_T^{\oplus n} \xrightarrow{\simeq} E_T\}. \end{aligned}$$

Show that Fr_E is representable by the open subscheme of

$$H := \mathbb{A}(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X^{\oplus n}, E)^\vee) = \mathbb{A}((E^\vee)^{\oplus n})$$

defined by $\det(u) \neq 0$, where $u : \mathcal{O}_H^{\oplus n} \to E_H$ is the universal homomorphism. Moreover, show that $\mathrm{Fr}_E \to X$ is a principal GL_n-bundle.

Recall that a morphism $f : X \to Y$ of schemes is *submersive* if f is surjective and Y has the quotient topology, i.e., a subset $U \subseteq Y$ is open if and only if $f^{-1}(U)$ is open, and $f : X \to Y$ is *universally submersive* if for every map $Y' \to Y$, the base change $X \times_Y Y' \to Y'$ is submersive.

Exercise 2.

(a) Show that a morphism $f : X \to Y$ of noetherian schemes is universally submersive if and only if every map $\mathrm{Spec}\ R \to Y$ from a DVR has a lift

$$\begin{tikzcd} \mathrm{Spec}\ R' \arrow[r, dashed] \arrow[d, dashed] & X \arrow[d, "f"] \\ \mathrm{Spec}\ R \arrow[r] & Y, \end{tikzcd}$$

where $R \to R'$ is a local homomorphism of DVRs.

(b) Show that a universally closed morphism of noetherian schemes is universally submersive.

(c) Show that every fpqc morphism of schemes is universally submersive.

Exercise 3 (Homework)**.**

(a) Let $P_1 \to X$ and $P_2 \to X$ be two principal G-bundles over X. Prove that any G-equivariant morphism $P_1 \to P_2$ over X is an isomorphism.

(b) Let X be a scheme and $d \geq 1$. Show that there is an equivalence of categories[1]

$$\begin{aligned}\{\text{finite, étale, and degree } d \text{ covers of } X\} &\xrightarrow{\sim} \{\text{principal } S_d\text{-bundles over } X\} \\ (Y \to X) &\mapsto (\underbrace{Y \times_X \cdots \times_X Y}_{d \text{ times}} \backslash \Delta \to X) \\ (P/S_{d-1} \to X) &\leftarrow\!\shortmid (P \to X).\end{aligned}$$

Here, the morphisms on the left-hand side are morphisms of X-schemes, and the morphisms on the right-hand side are S_d-equivariant morphisms over X. For the rightward map, the symmetric group S_d acts on the d-fold fiber product $Y \times_X \cdots \times_X Y$ by permutation, and Δ denotes the big diagonal, i.e., the S_d-equivariant closed locus of d-tuples where at least two points coincide. For the leftward map, P/S_{d-1} denotes the quotient of the free action by the subgroup $S_{d-1} \subseteq S_d$ fixing the dth index.

Exercise 4 (Bonus)**.** Let k be a field, k^{sep} a separable closure, and $G := \text{Gal}(k^{\text{sep}}/k)$ its Galois group with the profinite topology.

(a) Show that a covariant functor

$$F : \{\text{étale } k\text{-algebras}\} \to \text{Ab}$$

defines a sheaf of abelian groups on $(\operatorname{Spec} k)_{\text{ét}}$ if and only if $F(\prod_i A_i) = \bigoplus F(A_i)$ for every finite set of étale k-algebras $\{A_i\}$ and $F(K) = F(L)^{\text{Gal}(L/K)}$ for every finite Galois extension L/K of finite separable field extensions of k.

(b) Show that the category of sheaves on $(\operatorname{Spec} k)_{\text{ét}}$ is equivalent to the category of discrete G-modules, i.e., abelian groups with a continuous G-action.

Hint: Given F, define a discrete G-module by $M_F := colim_{k'/k} F(k')$, where the colimit is over finite separable field extensions. Conversely, given M, show that the functor on finite field extensions $k' \mapsto M^{Gal(k^{sep}/k')}$ extends to a sheaf F_M on $(\operatorname{Spec} k)_{\text{ét}}$. Show that $M \mapsto M_F$ and $M \mapsto F_M$ are inverse functors.

[1] That are actually *groupoids*.

Exercise 1 (Vector bundles vs GL-bundles)**.** For a scheme X, show that the assignment of a vector bundle E to the frame bundle Fr_E defines an equivalence of *groupoids*

$$\begin{aligned}\{\text{vector bundles over } X\} &\xrightarrow{\sim} \{\text{principal } \mathrm{GL}_n\text{-bundles over } X\}\\ E &\mapsto \mathrm{Fr}_E\end{aligned}$$

with the inverse defined by assigning $P \to X$ to the vector bundle whose total space is the quotient $P \times^{\mathrm{GL}_n} \mathbb{A}^n \coloneqq (P \times \mathbb{A}^n)/\mathrm{GL}_n$ of the diagonal GL_n-action on $P \times \mathbb{A}^n$.

Exercise 2 (Gluing sheaves)**.** Let $\mathcal{S}$ be a site and $(X_i \to X)$ be a covering in $\mathcal{S}$. If F_i are sheaves on the restricted sites $\mathcal{S}/X_i$ and $\alpha_{ij} : F_i|_{X_{ij}} \to F_j|_{X_{ij}}$ are isomorphisms of sheaves on $\mathcal{S}/X_{ij}$ (where $X_{ij} \coloneqq X_i \times_X X_j$) satisfying the cocycle condition $\alpha_{jk} \circ \alpha_{ij} = \alpha_{ik}$ on $\mathcal{S}/X_{ijk}$ (where $X_{ijk} \coloneqq X_i \times_X X_j \times_X X_k$), show that there exists a unique sheaf F on $\mathcal{S}/X$ and isomorphisms $\phi_i : F|_{X_i} \to F_i$ satisfying $\phi_j|_{X_{ij}} = \alpha_{ij} \circ \phi_i|_{X_{ij}}$.

Exercise 3. Given morphisms $\alpha : F \to H$ and $\beta : G \to H$ of presheaves on a category $\mathcal{S}$, the *fiber product* of α and β is the presheaf $F \times_H G$ whose set of sections over $S \in \mathcal{S}$ is $F(S) \times_{H(S)} G(S)$, i.e.,

$$\begin{aligned}F \times_H G : \mathcal{S} &\to \mathrm{Sets}\\ S &\mapsto \{(a,b) \in F(S) \times G(S) \mid \alpha_S(a) = \beta_S(b)\}.\end{aligned}$$

(a) Show that $F \times_H G$ is a fiber product in the category $\mathrm{Pre}(\mathcal{S})$.

(b) Show that if F, G, and H are sheaves on a site $\mathcal{S}$, then so is $F \times_H G$. In particular, $F \times_H G$ is also a fiber product in $\mathrm{Sh}(\mathcal{S})$.

Exercise 4 (Homework)**.** Let F be a presheaf on Sch. Show that the following are equivalent:

(a) F is a sheaf on $\mathrm{Sch}_{\text{ét}}$ (resp., $\mathrm{Sch}_{\mathrm{fppf}}$, $\mathrm{Sch}_{\mathrm{fpqc}}$).

(b) F sends coproducts to products (i.e., $F(\coprod_i U_i) = \prod_i F(U_i)$ for schemes U_i) and for every surjective étale (resp., fppf, faithfully flat) morphism $S' \to S$ of schemes, the sequence $F(S) \to F(S') \rightrightarrows F(S' \times_S S')$ is exact.

(c) F is a sheaf in the big Zariski topology $\mathrm{Sch}_{\mathrm{Zar}}$ and for every surjective étale (resp., fppf, faithfully flat) morphism $S' \to S$ of *affine* schemes, the sequence $F(S) \to F(S') \rightrightarrows F(S' \times_S S')$ is exact.

Hint: Given a covering $\{S_i \to S\}$, consider the map $\coprod_i S_i \to S$.

Exercise 1 (Groupoids)**.** After proving the following statements, upgrade them to the case of categories fibred in groupoids and prove the corresponding results.

(a) A groupoid $\mathcal{C}$ is equivalent to a set if and only if the diagonal functor $\Delta_{\mathcal{C}} \colon \mathcal{C} \to \mathcal{C} \times \mathcal{C}$ is fully faithful.

(b) Given two morphisms of groupoids $\mathcal{C}_1 \to \mathcal{S}$ and $\mathcal{C}_2 \to \mathcal{S}$, show that if $\mathcal{C}_1$ and $\mathcal{C}_2$ are equivalent to sets, then $\mathcal{C}_1 \times_{\mathcal{S}} \mathcal{C}_2$ is also equivalent to a set.

(c) For an action $\sigma \colon G \times S \to S$ in the category of sets, we have the following two Cartesian diagrams:

$$\begin{tikzcd} G \times S \arrow[r, "\sigma"] \arrow[d, "p_2"'] & S \arrow[d] \\ S \arrow[r] & {[S/G]} \end{tikzcd} \qquad \begin{tikzcd} G \times S \arrow[r, "{(\sigma, p_2)}"] \arrow[d] & S \times S \arrow[d] \\ {[S/G]} \arrow[r, "\Delta"] & {[S/G] \times [S/G]} \end{tikzcd}$$

Exercise 2 (Isom presheaves)**.** Let $\mathcal{X} \to \mathcal{C}$ be a category fibred in groupoids, and denote by $\mathcal{C}/S$ the restricted category over $S \in \mathcal{C}$.

(a) Show that for objects a and b of $\mathcal{X}$ over S that the functor

$$\begin{aligned} \underline{\mathrm{Isom}}_{\mathcal{X}(S)}(a,b) \colon \mathcal{C}/S &\to \mathrm{Sets} \\ (T \xrightarrow{f} S) &\mapsto \mathrm{Mor}_{\mathcal{X}(T)}(f^*a, f^*b), \end{aligned}$$

where f^*a and f^*b are choices of pullbacks, defines a presheaf on $\mathcal{C}/S$.

(b) Show that there is a Cartesian diagram

$$\begin{tikzcd} \underline{\mathrm{Isom}}_{\mathcal{X}(S)}(a,b) \arrow[r] \arrow[d] & S \arrow[d, "{(a,b)}"] \\ \mathcal{X} \arrow[r, "\Delta"] & \mathcal{X} \times \mathcal{X}. \end{tikzcd}$$

(c) Show that the presheaf $\mathrm{Aut}_{\mathcal{X}(S)}(a) = \underline{\mathrm{Isom}}_{\mathcal{X}(S)}(a,a)$ is naturally a presheaf of groups.

Exercise 3 (Homework)**.** Let $G \to S$ be a group scheme acting on a scheme U over S via $\sigma \colon G \times_S U \to U$, and let $[U/G]$ be the quotient stack. Let $T \to [U/G]$ be a morphism corresponding, via the 2-Yoneda lemma, to a principal G-bundle $P \to T$ and a G-equivariant map $f \colon P \to U$. Show that there is a Cartesian diagram

$$\begin{tikzcd} P \arrow[r, "f"] \arrow[d] & U \arrow[d] \\ T \arrow[r] & {[U/G].} \end{tikzcd}$$

Exercise 1 (Morphism gluing)**.** Let $\mathcal{X}$ be a category fibred in groupoids over a site $\mathcal{S}$. Show that Axiom (1) in the definition of stacks (morphisms glue, see class notes) is equivalent to the condition that for all objects a and b of $\mathcal{X}$ over $S \in \mathcal{S}$, the Isom presheaf $\underline{\mathrm{Isom}}_{\mathcal{X}(S)}(a, b)$ is a sheaf on the slice site $\mathcal{S}/S$ (cf. Ex. 4.2).

Exercise 2 (Generalizing Ex. 3.4)**.** Show that a category fibred in groupoids $\mathcal{X}$ over Sch is a stack over $\mathrm{Sch}_{\text{ét}}$ (resp., $\mathrm{Sch}_{\mathrm{fppf}}$, $\mathrm{Sch}_{\mathrm{fpqc}}$) if and only if $\mathcal{X}$ is a stack over $\mathrm{Sch}_{\mathrm{Zar}}$ and Axioms (1) and (2) in the definition of a stack (see class notes) hold for a surjective étale (resp., fppf, fpqc) morphism $\operatorname{Spec} A' \to \operatorname{Spec} A$ of affine schemes.

Exercise 3 (Universal family)**.** Let $g \geq 2$ be an integer. Using the same notation as in class, recall that $\mathcal{M}_{g,1}$ is the category fibred in groupoids whose objects over a scheme S are families of smooth curves $\mathcal{C} \to S$ together with a section $\sigma\colon S \to \mathcal{C}$. Let $\mathcal{M}_{g,1} \to \mathcal{M}_g$ be the morphism of categories fibred in groupoids that forgets the section. Show that if $S \to \mathcal{M}_g$ is a morphism corresponding to a family of curves $\mathcal{C} \to S$, then there is a Cartesian diagram:

$$\begin{tikzcd} \mathcal{C} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{M}_{g,1} \arrow[d] \\ S \arrow[r] & \mathcal{M}_g. \end{tikzcd}$$

Exercise 4 (Homework)**.** Consider an action of a group scheme G on a scheme X. For any G-invariant map $f\colon X \to Y$, define a natural morphism

$$\widetilde{f}\colon [X/G] \to Y.$$

Now consider the case where $G = \mathbb{G}_m$ acts by scalar multiplication on $X = \mathbb{A}^2 \setminus \{0\}$ and let $f\colon \mathbb{A}^2 \setminus \{0\} \to \mathbb{P}^1$ be the standard morphism given by $(x, y) \mapsto [x : y]$. Prove that the induced map $\widetilde{f}$ is an isomorphism. Is $\widetilde{f}$ still an isomorphism if instead the action of $\mathbb{G}_m$ is given by $t \cdot (x, y) = (t^2x, t^2y)$?

Definition (Induced G-bundles)**.** Let $H \to G$ be a homomorphism of fppf affine groups schemes over a scheme S. If $P \to X$ is a principal H-bundle, the *principal G-bundle induced by P via $H \to G$* is

$$G \times^H P \to X,$$

where $G \times^H P$ is the quotient $(G \times P)/H$ of the action $h \cdot (g,p) = (gh^{-1}, hp)$, and where G acts on $G \times^H P$ via $g' \cdot (g,p) = (g'g, p)$.

Exercise 1. Show that stackification commutes with fiber products: if $\mathcal{X} \to \mathcal{Y}$ and $\mathcal{Z} \to \mathcal{Y}$ are morphisms of prestacks, then $(\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z})^{\mathrm{st}} \cong \mathcal{X}^{\mathrm{st}} \times_{\mathcal{Y}^{\mathrm{st}}} \mathcal{Z}^{\mathrm{st}}$.

Exercise 2.

(a) If $H \to G$ is a morphism of smooth affine group schemes over a scheme S, define a morphism of stacks $BH \to BG$ over Sch/S by using induced G-bundles to construct a principal G-bundle from a principal H-bundle. Finally, show that

$$BH \times_{BG} S \cong [G/H].$$

(b) If $1 \to K \to G \to Q \to 1$ is an exact sequence of smooth affine algebraic groups over a field k, show that there is a cartesian diagram

$$\begin{tikzcd} Q \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & BK \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \operatorname{Spec} k \arrow[d] \\ \operatorname{Spec} k \arrow[r] & BG \arrow[r] & BQ. \end{tikzcd}$$

Exercise 3. Let $V = \operatorname{Spec} k[x,y]/(xy)$ and $U = V \bigcup_{V \setminus 0} V$ be the non-separated union. Let $\mathbb{Z}/2$ act on U exchanging the branches and swapping the origins, and let $X = U/(\mathbb{Z}/2)$ be the quotient algebraic space.

The affine line with a nodal singularity at the origin.

(a) Show that the diagonal of X is not a locally closed immersion, and conclude that X is not a scheme.

(b) Show that X has dimension one with a singularity at the origin, and that there is a universal homeomorphism $X \to \mathbb{A}^1$ which is an isomorphism away from the origin.

Exercise 4 (Homework)**.** Let $G \to S$ be a smooth affine group scheme acting on a scheme $U \to S$.

(a) Show that $[U/G]^{\mathrm{pre}}$ satisfies Axiom (1) of a stack over $(\mathrm{Sch}/S)_{\text{ét}}$.

(b) Show that $[U/G]$ is isomorphic to the stackification of $[U/G]^{\mathrm{pre}}$ over $(\mathrm{Sch}/S)_{\text{ét}}$, and that $[U/G]^{\mathrm{pre}} \to [U/G]$ is fully faithful.

Exercise 1. For a field k, define $\mathrm{Gr}(q,n)_k := \mathrm{Gr}(q,n) \times_{\mathbb{Z}} k$, and let $p \in \mathrm{Gr}(q,n)_k$ be a point corresponding to $W \subset k^n$ with quotient $Q := k^n/W$. Show that there is a natural bijection of the tangent space

$$\mathrm{T}_p(\mathrm{Gr}(q,n)_k) \xrightarrow{\sim} \mathrm{Hom}_k(W,Q)$$

with the vector space of k-linear maps $W \to Q$.

Exercise 2. Use the valuative criterion of properness to show that $\mathrm{Gr}(q,n) \to \operatorname{Spec}\mathbb{Z}$ is proper.
Optional: Let S be a noetherian scheme and F be a coherent sheaf on $\mathbb{P}^n_S$ that can be written as a quotient of $\mathcal{O}_{\mathbb{P}^n_S}(l)^{\oplus r}$ for some l and r. Fix a polynomial $p \in \mathbb{Q}[z]$. Prove that the functor $\mathcal{Q}uot(F/\mathbb{P}^n_S/S)$ is proper over S using the valuative criterion.

Definition (Strongly projective)**.** A morphism $f : X \to S$ of schemes is said to be strongly projective if there exists a locally free $\mathcal{O}_S$-module $\mathcal{E}$ of finite rank such that f factors as a closed immersion followed by the natural projection:

$$X \hookrightarrow \mathbb{P}(\mathcal{E}) \to S$$

where $\mathbb{P}(\mathcal{E})$ denotes the relative projective space associated to $\mathcal{E}$.

Exercise 3.

a) Let X be projective over a field k. Modifying the argument for the case of curves, prove that $\mathcal{B}un(X)$ and $\mathcal{C}oh(X)$ are algebraic stacks.

b) More generally, show that if $X \to S$ is a strongly projective morphism of noetherian schemes, then the stack $\mathcal{C}oh(X/S)$, whose objects over an S-scheme T are finitely presented, quasi-coherent sheaves on X_T flat over T, is an algebraic stack.

Exercise 4 (Homework, Flag varieties)**.** Fix a positive integer n and a decreasing sequence $n > q_1 > \cdots > q_s > 0$ of integers. Show that the functor Sch $\to$ Sets, taking a scheme S to the set of isomorphism classes of sequences of quotients

$$\mathcal{O}_S^{\oplus n} \twoheadrightarrow V_1 \twoheadrightarrow \cdots \twoheadrightarrow V_s,$$

where each V_i is a vector bundle on S of rank q_i, is representable by a projective scheme over $\mathbb{Z}$.

Exercise 1. Let $\mathcal{X}$ be a Deligne–Mumford stack.

(a) Show that the diagonal of $\mathcal{X}$ is unramified.

(b) For a field-valued point $x \in \mathcal{X}(k)$, show that G_x is a separated étale group scheme over k. If in addition G_x is quasi-compact, show that G_x is a finite étale group scheme over k.

Hint: First show that G_x is an étale group algebraic space over k which becomes a scheme after the base change by a finite field extension $k \subset k'$. Apply the Descent Criterion for an Fppf Sheaf (Prop. 3.3.17 in Alper Book) to conclude that G_x is a scheme.

Exercise 2.

(a) If G is a smooth affine algebraic group over a field k, show that the inertia stack of BG is the quotient $[G/G]$ where G acts on itself via conjugation. In particular, if G is abelian then $I_{BG} \cong G \times BG$.

(b) More generally, show that if G acts on a k-scheme U, show that $I_{[U/G]} \cong [J/G]$ where $J = \{(h,u) \in G \times U \mid h \cdot u = u\}$ and the action is given by $g \cdot (h,u) = (ghg^{-1}, gu)$.

(c) Let G be a group scheme over k corresponding to a finite abstract group. If G acts on a k-scheme U, then

$$I_{[U/G]} = \coprod_{g \in \mathrm{Conj}(G)} [U^g / C_g],$$

where $\mathrm{Conj}(G)$ is the set of conjugacy classes of elements of G, C_g is the centralizer of g, and $U^g := \{x \in U \mid gx = x\}$ is the closed locus fixed by g.

Exercise 3. If $U \to \mathcal{X}$ is a smooth presentation of an algebraic stack, show that $|U| \to |\mathcal{X}|$ is submersive, i.e., is surjective and $|\mathcal{X}|$ has the quotient topology.
Hint: If $V \subseteq |\mathcal{X}|$ is a subset whose preimage U' in U is open, define $\mathcal{V} \subseteq \mathcal{X}$ as the substack of objects $T \to \mathcal{X}$ such that $U'_T = U_T$.

Exercise 4 (Homework)**.**

(a) If $\mathcal{X} \to \mathcal{Y}$ is a representable morphism of algebraic stacks (e.g., a morphism of algebraic spaces), show that $\mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ is representable by schemes.

(b) If $\mathcal{X} \to \mathcal{Y}$ is a morphism of algebraic stacks, show that $\mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ is representable.

Exercise 1. Show that if $\mathcal{X}$ is an algebraic stack (resp., algebraic space) and $U \to \mathcal{X}$ is a smooth presentation, then $\mathcal{X}$ is isomorphic to the quotient stack $[U/R]$ (resp., quotient sheaf U/R) of the smooth groupoid (resp., equivalence relation) $R \rightrightarrows U$ where $R = U \times_{\mathcal{X}} U$.

Exercise 2. Let x be a k-point of an algebraic stack $\mathcal{X}$. Show that $T_{\mathcal{X},x}$ is naturally a representation of G_x which is given set-theoretically by: $g \cdot (\tau, \alpha) = (\tau, g \circ \alpha)$ for $g \in G_x$ and $(\tau, \alpha) \in T_{\mathcal{X},x}$.

Exercise 3. Let k be a field. If $x\colon \operatorname{Spec} k \to B\mu_n$ denotes the canonical presentation, compute the tangent space $T_{B\mu_n,x}$.

Hint: The answer depends on the characteristic.

Exercise 4 (Homework)**.** Extend Exercise 4.1 to show that if $s, t\colon R \rightrightarrows U$ is a smooth groupoid of algebraic spaces, the following diagrams are cartesian:

$$\begin{tikzcd} R \arrow[r, "s"] \arrow[d, "t"'] \arrow[dr, phantom, "\square"] & U \arrow[d, "p"] \\ U \arrow[r, "p"'] & {[U/R]} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} R \arrow[r, "{(s,t)}"] \arrow[d] \arrow[dr, phantom, "\square"] & U \times U \arrow[d, "p \times p"] \\ {[U/R]} \arrow[r, "\Delta"'] & {[U/R] \times [U/R]} \end{tikzcd} .$$

Exercise 1 (Proper actions)**.** An action of an algebraic group G over an algebraically closed field k on an algebraic space U is called *proper* if the action map

$$\Psi : G \times U \to U \times U, \quad (g, u) \mapsto (gu, u)$$

is proper.

(a) Show that the action of G on U is proper if and only if $[U/G]$ is separated.

(b) For $u \in U(k)$, let $\Psi_u : G \to U$ be the map defined by $g \mapsto gu$ (viewing Ψ as a morphism over U via the projections on the second components, then Ψ_u is the fiber of Ψ over u). Show that the following are equivalent:

(i) $\Psi_u : G \to U$ is proper,

(ii) $u : \operatorname{Spec} k \to [U/G]$ is proper,

(iii) $Gu \subseteq U$ is closed and G_u is proper.

Hint for (b): To show that (i) or (ii) implies (iii), replace U with the reduced orbit $(Gu)_{\mathrm{red}}$, use Generic Flatness to show that the morphisms Ψ_u or u are faithfully flat, and then use fppf descent.

Exercise 2. Let $\mathcal{X}$ be a noetherian algebraic stack and $x \in |\mathcal{X}|$ be a finite type point with smooth stabilizer. Let $\bar{x}\colon \operatorname{Spec} k \to \mathcal{X}$ be a representative of x. Show that

$$\dim \mathcal{G}_x = -\dim G_{\bar{x}}.$$

Exercise 3. Use the valuative criterion to show the following:

(a) If G is an abstract finite group, then $BG \to \operatorname{Spec} \mathbb{Z}$ is proper.

(b) The map $B\mathbb{G}_m \to \operatorname{Spec} \mathbb{Z}$ is universally closed but not separated.

Exercise 4 (Homework)**.** Using the same notation as in class, recall that $\mathcal{M}_{g,n}$ is the algebraic stack whose objects over a scheme S are families of smooth genus g curves $\mathcal{C} \to S$ together with n sections $\sigma_i\colon S \to \mathcal{C}$, for $1 \leq i \leq n$.

Show that $\mathcal{M}_{0,0}$, $\mathcal{M}_{0,1}$, $\mathcal{M}_{0,2}$, and $\mathcal{M}_{1,0}$ are not separated.

Exercise 1. Let k be a field, and $d_0, \dots, d_n$ be positive integers. Denote by $\mathcal{P}(d_0, \dots, d_n)$ over $\operatorname{Spec} k$ the weighted projective stack as defined in class.

(a) If k is a field of characteristic p, show that $\mathcal{P}(d_0, \dots, d_n)$ is a Deligne–Mumford stack over k if and only if p does not divide any d_i.

(b) Show that there is a bijective morphism to the weighted projective space

$$\pi \colon \mathcal{P}(d_0, \dots, d_n) \to \operatorname{Proj} k[x_0, \dots, x_n],$$

where $\deg(x_i) = d_i$. Prove that π is a coarse moduli space.

Exercise 2. Let X be a scheme and r be an integer invertible in $\Gamma(X, \mathcal{O}_X)$. Fix a line bundle L on X and a section $s \in \Gamma(X, L)$.

(a) Show that the root stack $X(\sqrt[r]{L, s})$ is a Deligne–Mumford stack.

(b) Show that $X(\sqrt[r]{L, s})$ is equivalent to the category of triples $(f \colon T \to X, M, \alpha, t)$, where $f \colon T \to X$ is a morphism of schemes, M is a line bundle on T, $\alpha \colon M^{\otimes r} \to f^*L$ is an isomorphism, and $t \in \Gamma(T, M)$ is a section such that $\alpha(t^{\otimes r}) = f^*s$. In particular, show that there is a line bundle $L^{1/r}$ on $X(\sqrt[r]{L, s})$ with a section $s^{1/r}$ together with an isomorphism $(L^{1/r})^{\otimes r} \xrightarrow{\sim} \pi^*L$ identifying $(s^{1/r})^{\otimes r}$ with π^*s.

(c) If $X = \operatorname{Spec} A$ is an affine scheme and $L = \mathcal{O}_X$ is trivial, show that

$$X(\sqrt[r]{\mathcal{O}_X, s}) \cong [\operatorname{Spec}(A[x]/(x^r - s))/\mu_r],$$

where μ_r acts on $\operatorname{Spec}(A[x]/(x^r - s))$ via $t \cdot x = tx$.

(d) Show that the morphism $X(\sqrt[r]{L, s}) \to X$ is a coarse moduli space and an isomorphism over the open subscheme $X_s = \{x \in X \mid s_x \neq 0\}$.

Note: The hypothesis that $r \in \Gamma(X, \mathcal{O}_X)^*$ ensures that the group scheme $\mu_{r,X} \to X$ is étale.

Exercise 3 (Homework)**.** Let k be a field. Using the notation from Exercise 1:

(a) Classify all points of $\mathcal{P}(3, 3, 4, 6)$ that have non-trivial stabilizers.

(b) An algebraic stack $\mathcal{X}$ is said to have *generically trivial stabilizer* if there exists a dense open substack $U \subset \mathcal{X}$ which is an algebraic space. Provide conditions under which $\mathcal{P}(d_0, \dots, d_n)$ has a generically trivial stabilizer.

Exercise 1. Let X be a scheme and $\mathbb{G}_m$ be the multiplicative group over $\operatorname{Spec}\mathbb{Z}$. Show that a quasi-coherent sheaf on $X \times B\mathbb{G}_m$ is the same data as a $\mathbb{Z}$-graded quasi-coherent sheaf on X.
Hint: Extend the argument used to classify $\mathbb{G}_m$-representations.

Exercise 2. Let $f\colon \mathcal{X} \to \mathcal{Y}$ be an affine morphism of noetherian algebraic stacks.

(a) Show that if $\mathcal{Y}$ is Θ-complete, so is $\mathcal{X}$.

(b) Assume that $\mathcal{Y} \to Y$ is a good moduli space. Show that for every field valued point $p \in \mathcal{Y}(k)$ whose image in $|\mathcal{Y}|$ is a closed point, the stabilizer G_p is linearly reductive.

Hint for (a): Use the fact that $0 \in \Theta_R$ has codimension two.

Exercise 3. Let G be a smooth linearly reductive group over an algebraically closed field k, and let U be a separated algebraic space of finite type over k with an action of G.

(a) Prove that $B\,\mathrm{GL}_n$ is Θ-complete. Deduce then that BG is Θ-complete using Matsushima's characterization of reductive subgroups (See Thm. B.1.48 in Alper Book).

(b) Prove that $[U/G]$ is Θ-complete if and only if for every one-parameter subgroup $\lambda\colon \mathbb{G}_m \to G$, the morphism
$$\begin{aligned} \mathrm{ev}_1\colon U_\lambda^+ := \underline{\mathrm{Mor}}_k^{\mathbb{G}_m}(\mathbb{A}^1, U) &\longrightarrow U \\ f &\longmapsto f(1) \end{aligned}$$
is a closed immersion.

(c) Conclude that if $U = \operatorname{Spec} A$ is affine, then $[U/G]$ is Θ-complete.

Hint for (a): Use again the fact that $0 \in \Theta_R$ has codimension two.
For (b): Assume that U_λ^+ is representable by an algebraic space, and ev_1 is a monomorphism.
Remark for (c): Conclude either from the condition in (b), or directly from (a).

Exercise 4 (Homework)**.** Let G be an affine algebraic group over a field k, and consider an affine k-scheme $\operatorname{Spec} A$ with a G-action. Recall that a G-representation is a k-vector space with a dual action $\sigma\colon V \to \Gamma(G, \mathcal{O}_G) \otimes_k V$ satisfying two natural compatibility conditions.

(a) Show that a quasi-coherent sheaf on $[\operatorname{Spec} A/G]$ is the data of an A-module M together with a coaction $\sigma\colon M \to \Gamma(G, \mathcal{O}_G) \otimes_k M$ over k (i.e., a map of k-vector spaces giving M the structure of a G-representation) such that the multiplication map $A \otimes_k M \to M$ is a morphism of G-representations.

(b) Considering the diagram
$$\begin{tikzcd} \operatorname{Spec} A \arrow[r, "p"] & {[\operatorname{Spec} A/G]} \arrow[r, "\pi"] \arrow[d, "q"] & \operatorname{Spec} A^G \\ & BG & \end{tikzcd},$$
provide descriptions of the functors $p_*, p^*, \pi_*, \pi^*, q_*$, and q^* on quasi-coherent sheaves.