

Seminar on Lie Algebras and Representation Theory

Winter Semester 2026/2027 — University of Duisburg-Essen

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1 Description

Lie algebras arise naturally in the study of symmetries of mathematical objects and play a fundamental role in algebra and geometry. Roughly speaking, they capture the “infinitesimal behavior” of group actions, allowing us to study symmetries through the more linear framework of algebra. This perspective brings powerful tools from linear algebra and combinatorics into play and leads to the elegant classification of semisimple Lie algebras, which we will develop throughout this seminar. To any algebraic (or Lie) group G , we can functorially associate a finite-dimensional Lie algebra $\mathfrak{g}$. In the final lectures, we will explore the relationship between groups and their associated Lie algebras. We will conclude by emphasizing how the representation category $\text{Rep}(G)$ can be understood through the corresponding category $\text{Rep}(\mathfrak{g})$.

Schedule: Tuesday, 16–18 in room WSC-N-U-4.04.

Prerequisites: A first course in the representation theory of finite groups is recommended but not necessary. Towards the end, we will assume a bit of familiarity with affine algebraic varieties. In particular, we will use the tangent bundle, which will be recalled in one of the last talks.

Guidelines for the talks

We will develop the theory over an algebraically closed field of characteristic zero. For Lie algebras, we will mainly use [Hum72], while for the last part we will use both [Mil18] and [Hum75].

- The talks are given in English at the blackboard and should last approximately 80 minutes, to allow for 10 minutes of questions.
- Participants are expected to discuss their talk with the lecturer the week before they are scheduled to speak, bringing notes for their talk (handwritten is fine) at the end of the previous seminar.
- All terms introduced in italics in the descriptions below should be defined.
- In every single talk description below, we suggest a list of checkpoints to consider for the journey of your talk. In order to explain the material, the use of examples is encouraged.

2 Talks

Talk 1: Overview and Organization (Lecturer)

After providing motivation and a brief overview of the seminar’s topics, we will finalize the schedule and distribute the remaining talks among the participants.

Talk 2: Introduction to Lie Algebras (Lecturer)

Definition of *Lie algebras*, *(mono/epi)morphisms* of Lie algebras, ideals, subalgebras, and quotients. Examples: $\mathfrak{gl}(V)$, $\mathfrak{t}$, $\mathfrak{n}$, $\mathfrak{d}$, $\mathfrak{sl}(V)$, $\mathfrak{sp}(V)$, $\mathfrak{o}(V)$. Homomorphism theorems for Lie algebras. Classification in dimensions 0 and 1. Definition of *center*, *derived algebra*, *simple algebra*, and *indecomposable and irreducible representations* of a Lie algebra. Definition of the adjoint representation and derivations $\text{Der } \mathfrak{U} \subset \text{End } \mathfrak{U}$. Adjoint representation $L \xrightarrow{\text{ad}_L} \text{Der } L$ and the center $Z(L)$. Simple Lie algebras are linear; statement of Ado’s Theorem.

Talk 3: Semisimple Lie Algebras

Define the *derived series* and *solvable Lie algebras*, and provide examples. After proving [Hum72, Prop. 3.1], define the *radical* and *semisimple Lie algebras*. Definition of *nilpotent* and *ad-nilpotent*

elements, and [Hum72, Prop. 3.2]. State and prove Engel's theorem and the existence of a flag for a Lie algebra whose elements are nilpotent; follow the exposition in [Hum72, Sec. 3.2, 3.3].

Reference: [Hum72, Sec. 3].

Talk 4: Lie's Theorem, Jordan Decomposition, and Cartan's Criterion

State and prove Lie's theorem (state also the version about a flag), apply it to show that $[L, L]$ is nilpotent whenever L is solvable and that $\mathfrak{sl}_n$ is semisimple. State the Jordan-Chevalley decomposition (without proof) and prove that it is ad-stable. State and prove Cartan's criterion and its converse; notice that it is enough to verify $\text{Tr}(xy) = 0$ on a basis ([Hum72, Ex. 4.7, 4.8]).

Reference: [Hum72, Sec. 4].

Talk 5: Killing Form and Semisimplicity

Define the *Killing form*, and compute it explicitly for $\mathfrak{sl}_2$. Prove that a Lie algebra is semisimple if and only if its Killing form is non-degenerate. Use the form to prove the decomposition of semisimple Lie algebras and that all derivations are inner for semisimple Lie algebras. Abstract Jordan decomposition. Show that $x \in \mathfrak{gl}_n$ is semisimple if and only if $\text{ad } x$ is.

Reference: [Hum72, Sec. 5].

Talk 6: Lie Algebra Representations

Recall the definition of an L -module, define duals, tensor products of representations, and $\text{End}(V)$ as an L -module (whose invariants are module homomorphisms). Define *irreducible* and *indecomposable representations*, state and prove Schur's lemma. Define the Casimir element and prove that representations of a semisimple Lie algebra are completely reducible (Weyl's theorem), i.e., $\text{Rep } L$ is a semisimple category. Define *reductive Lie algebras* and prove the converse of Weyl's Theorem using [Hum72, Ex. 6.5(a,c,d)]. Prove that the abstract Jordan decomposition is compatible with linear representations (and therefore abstract and usual Jordan decompositions coincide).

Reference: [Hum72, Sec. 6].

Talk 7: Representations of $\mathfrak{sl}_2$

Recall the Lie algebra $\mathfrak{sl}_2$ and define the *weight space decomposition* for an $\mathfrak{sl}_2$ -module. Classify (irreducible) representations of $\mathfrak{sl}_2$, realizing them as in [Hum72, Ex. 7.4]. Example: decompose $\mathfrak{sl}_3$ as an $\mathfrak{sl}_2$ -module (via the upper-left hand inclusion), cf. [Hum72, Ex. 7.2]. Construct an irreducible infinite-dimensional representation [Hum72, Ex. 7.7].

Reference: [Hum72, Sec. 7].

Talk 8: Cartan Decomposition and Root Systems

Define *toral subalgebras* and prove that they are abelian. Introduce the *Cartan decomposition* by showing [Hum72, Prop. 8.2] and the non-degeneracy of the Killing form restricted to L_0 . Prove that for maximal toral subalgebras H we have $H = C_L(H)$. Define a *root system* and, through the results of [Hum72, Sec. 8.3, 8.4], prove that we can associate to any choice of maximal toral subalgebra $H \subset L$ a root system (Φ, E) whose Euclidean scalar product is given by the Killing form. Discuss how the Cartan decomposition gives inclusions $\mathfrak{sl}_2 \subset L$. Example: the root system of $\mathfrak{sl}_n$.

Reference: [Hum72, Sec. 8].

Talk 9: Root System Classification

Define *reflections*, *Weyl groups*, and *simple roots*. Introduce the *Cartan matrix* and show that roots can have at most two different lengths. Define *Coxeter graphs* and *Dynkin diagrams*. State the classification of *irreducible* root systems via Dynkin diagrams, and overview the techniques used in the proof. Provide examples of root systems throughout the talk.

Main reference: [Hum72, Sec. 9, 10, 11].

Talk 10: Isomorphism Theorem and Engel Subalgebras

Show that simple Lie algebras correspond to irreducible root systems and prove the isomorphism theorem [Hum72, Thm. 14.2]. Discuss automorphisms that come from the Weyl group [Hum72, Sec. 14.3, Ex. 14.5]. Define *Engel subalgebras* and prove Lemmas A and B in [Hum72, Sec. 15.2].

Reference: [Hum72, Sec. 14, 15.1, 15.2].

Talk 11: Cartan Subalgebras and Conjugacy Theorem

Define *Cartan subalgebras* (CSAs) and prove that minimal Engel subalgebras are CSAs. Functorial properties of CSAs. Define $\mathcal{E}(L) \subset \text{Aut}(L)$ and prove the Conjugacy Theorem for solvable Lie algebras. Define *Borel subalgebras* and prove that these are conjugate under $\mathcal{E}(L)$ in any Lie algebra. Conclude that CSAs are conjugate and hence, for semisimple Lie algebras, root systems associated to distinct maximal toral subalgebras are isomorphic. Conclude by stating that for any root system there exists a Lie algebra realizing it (Serre's Theorem); this concludes the classification of finite-dimensional Lie algebras.

Reference: [Hum72, Sec. 15.3, 15.4, 16].

Talk 12: Universal Enveloping Algebra and Representations

Define the *tensor and symmetric algebras*. Introduce the *universal enveloping algebra* via its universal property, and establish its existence and uniqueness. State the Poincaré-Birkhoff-Witt (PBW) Theorem (along with [Hum72, Cor. A, B, C, D]) and sketch the proof. State the equivalence between L -modules and representations of the algebra $\mathfrak{U}(L)$. Recall weight spaces and define a *maximal vector* for a representation. Define *standard cyclic* representations and prove the characterization given in [Hum72, Thm. 20.2]. State and prove the existence and uniqueness of *standard cyclic modules*, and show how this yields a classification of finite-dimensional representations.

Reference: [Hum72, Sec. 17, 20].

Talk 13: Lie Algebra of an Algebraic Group in Characteristic Zero

Define the notion of an *affine algebraic group* $G/\mathbb{C}$ and G -representation. After recalling the notion of tangent space, construct the functor $\text{Lie}: G \mapsto \mathfrak{g}$ associating a Lie algebra to a group G , and define $\text{Ad}: G \rightarrow \text{GL}(\mathfrak{g})$. Show that the derivative of Ad is ad . Consider the examples $\text{GL}_n, \text{SL}_n, \mathbb{G}_a$. Show that a map between algebraic groups induces a map on the corresponding Lie algebras, cf. [Mil18, Thm. II.3.8], [Hum75, Thm. 9.1]. Prove properties [Mil18, II.3.19, II.3.20, II.3.21] (cf. also [Hum75, 12.5]). Show that Lie is exact and faithful, but not full (providing a counterexample). Construct the functor $\text{Rep } G \rightarrow \text{Rep } \mathfrak{g}$ using Lie and prove it is fully faithful, cf. [Mil18, pp. 131–133]. Prove the correspondence between normal subgroups of connected algebraic groups and ideal subalgebras ([Hum75, Thm. 13.2]), and the correspondence for $N_G(W)$ and $C_G(W)$. Show that there is no general correspondence between subgroups of G and Lie subalgebras of $\mathfrak{g}$, providing an example of a subalgebra that does not come from a subgroup.

Reference: [Mil18, Sec. II.3], see also [Hum75, Sec. 9.1].

Talk 14: Panorama on Semisimple and Reductive Groups (open)

There are few possibilities, find in the following line possible topics. Define the notion of *semisimple* and the more general notion of *reductive* algebraic groups via the *radical* and *unipotent radical*. Prove that a connected group G is semisimple if and only if $\text{Lie}(G)$ is semisimple, and as a corollary prove that $\text{Rep } G$ is semisimple [Hum75, Sec. 14.3]. Prove that G reductive implies $\text{Lie}(G)$ reductive, but the converse does not hold (e.g., $\mathbb{G}_a$). As an example, prove that SL_n and PGL_n are semisimple (with the same Lie algebra $\mathfrak{sl}_n$), while GL_n is not.

Prove that $\text{Rep } G$ is semisimple if and only if the functor $V \rightarrow V^G$ is exact; such groups are called *linearly reductive*. Prove that affine linearly reductive groups are reductive (cf. [Hum75, Thm. 17.5]), and prove that the converse is true in characteristic 0 (providing an example that fails if $\text{char } k > 0$).

Prove that any action of a linearly reductive group on an affine variety has a finitely generated ring of invariants (Hilbert's 14th Problem).

Overview the classification of reductive groups via root data.

Reference: [Mil18, Sec. II.3], see also [Hum75, Sec. 13.5, 14.3].

References

- [Hum72] James E Humphreys. Introduction to lie algebras and representation theory. 1972.
- [Hum75] James E Humphreys. *Linear algebraic groups*, volume 21. Springer New York, 1975.
- [Mil18] James S. Milne. Lie algebras, algebraic groups, and lie groups. Course notes, 2018. Available at <https://www.jmilne.org/math/CourseNotes/LAG.pdf>.