Seminar on Toric Varieties

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1 Description

Toric varieties form a special and important class of algebraic varieties. These are varieties that contain a dense algebraic torus whose group action extends to the whole variety. This condition allows for explicit constructions from combinatorial data (like lattices, fans, polytopes) and establishes a precise correspondence between toric varieties and purely combinatorial structures. This gives rise to an elegant dictionary translating geometric properties into purely combinatorial terms. As a result, toric varieties provide a fertile source of (counter)examples in algebraic geometry and are extremely useful for developing intuition for more abstract notions and constructions, such as those arising in singularity theory and GIT.

Schedule: Wednesday 14–16, Tea Room (to be discussed at the first meeting).

Prerequisites: The main prerequisite is a bit of familiarity with Spec A and with gluing constructions. A first course on schemes is recommended (possibly in parallel with this one).

Guidelines for the talks

We will mainly follow [Ful93, CLS11, Nab]. These contain many exercises, and you should engage with them.

- The talks are given in English at the blackboard and should last approximately 80 minutes, to allow for 10 minutes of questions.
- Participants are expected to discuss their talk with the lecturer the week before they are scheduled to speak, bringing notes of their talk (handwritten is fine) at the end of the previous seminar.
- All the definitions of terms in italics in the descriptions below should be given.
- In every single talk description below, we suggest a list of check-points to consider for the journey of your talk. In order to explain the material, the use of examples is encouraged.

2 Talks

Talk 1: Overview and Organization (15/04/26, D. Gori)

After providing motivation and a brief overview of the seminar's topics, we will finalize the schedule and distribute the remaining talks among the participants.

Talk 2: Preliminaries (22/04/26, D. Gori)

We recall some basic constructions in algebraic geometry assumed for this course, including the gluing of schemes and the theory of group actions (specifically tori and their (co)characters). We will also introduce Spec $\mathbb{C}[S]$ for a semigroup S.

Talk 3: Rational Cones (29/04/26)

Define the notions of *lattice*, *rational convex polyhedral cone*, *face*, and the *dimension* of a cone. Discuss the fundamental properties of convex polyhedral cones following [Ful93, Sec. 1.2]. Introduce the *dual cone* and state Gordon's Lemma (without proof). Show that cones can be described as intersections of half-spaces and explain how to pass between the generator and the constraint descriptions, providing explicit examples.

Main references: [Ful93, Sec. 1.2], [Nab, Sec. 4.1]; see also [Pega, Section on cones].

Talk 4: Affine Toric Varieties (05/05/26)

Define *affine toric varieties* via semigroups as in [Ful93, Sec. 1.3], describing $\mathbb{C}$-points and morphisms of affine toric varieties. Describe the association $\sigma \mapsto \operatorname{Spec} \mathbb{C}[S_\sigma]$, the embedding of the torus via lattices, and the geometric interpretation of face relations. Use [Tel22, Prop. 2.64] to upgrade this association to a functor between the category of cones σ and affine toric varieties containing the torus $\operatorname{Spec} \mathbb{C}[M]$. Provide examples throughout the talk, considering maps induced by sublattices of smaller dimension and coarsening of the lattice.

Main reference: [Ful93, Sec. 1.3]; see also [Pega, Affine toric varieties].

Talk 5: Semigroups and Equivalent Definitions (12/05/26)

Define the *groupification* of a semigroup and define the properties *integral, torsion-free,* and *affine* for semigroups, providing examples and counterexamples. State and prove [Nab, Thm. 4.15]. Furthermore, define *saturated* semigroups and prove [Nab, Thm. 4.17] and [Nab, Lemma 4.21]. Conclude the first part by stating the equivalence of definitions for normal affine toric varieties. Afterwards, define $Y_{\mathscr{A}}$ ([CLS11, Def. 1.1.7]) and *toric ideals*. Show that these correspond to toric varieties, and state and prove the equivalence in [CLS11, Thm. 1.1.17]. Provide an explicit example of an affine toric variety expressed in all three equivalent ways (semigroup, algebraic, and via toric ideals).

Main references: [Nab, From Def. 4.9 to the end of Chap. 4], [CLS11, Sec. 1.1]; see also [Pegb, Toric ideals].

Talk 6: Toric Varieties from Polyhedral Fans (19/05/26)

Define *rational polyhedral fans* and the associated normal toric varieties $X(\Delta)$ via the gluing of affine patches. Provide examples from [Ful93, Sec. 1.4], as well as weighted projective spaces [CLS11, Ex. 3.1.16, 3.1.17]. Prove that $X(\Delta)$ is separated. Identify the natural action of the maximal torus on $X(\Delta)$. Following the exposition in [Tel22, Sec. 4.3], prove the characterization of limits of one-parameter subgroups, explaining how these limits recover the data of the fan. Illustrate these limits using the previously introduced examples.

Main references: [Ful93, Sec. 1.4], [CLS11, Sec. 3.1], [Tel22, Sec. 4.3]. See also [Pegb, Toric Varieties from Polyhedral Fans].

Talk 7: The Orbit-Cone Correspondence (26/05/26)

Recall the description of closed points in toric varieties. State and prove the *orbit-cone correspondence* theorem following [Tel22, Sec. 4.4]. State Sumihiro's Theorem [CLS11, Thm. 3.1.7] (without proof) and use it to prove that any normal toric variety is of the form $X(\Delta)$. Define *toric morphisms* as in [Tel22, Def. 4.32], and upgrade the correspondence between fans and toric varieties to an equivalence of categories, following [Tel22, Sec. 4.6]. Exhaust the classification in dimension 1, and provide the example of weighted projective spaces in dimension 2.

Main references: [Tel22, Sec. 4.4, 4.6], [CLS11, Ex. 3.2.11]. See also: [Pegb, The Orbit-Cone Correspondence].

Talk 8: Smoothness and Properness

Recall the notion of *distinguished points* and characterize the smoothness of (affine) toric varieties, stating and proving the first proposition of [Ful93, Sec. 2.1]. Prove the characterization of proper toric maps (eventually over a point) and provide the example of blow-ups following [Ful93, Sec. 2.4]. State and prove that any smooth proper toric variety of dimension two is obtained as a sequence of blow-ups of $\mathbb{P}^2$ or $\mathbb{F}_a$. Before treating the toric case, briefly recall the general notions of smoothness and the construction of a blow-up.

Main reference: [Ful93, Sec. 2.1, 2.4, 2.5]; see also [Tel22, Sec. 4.6.3], [Nab, Sec. 9.1.2].

Talk 9: Toric Surfaces and Resolution of Singularities

Recall basic notions of quotients of affine varieties by finite groups, motivating the quotient $\operatorname{Spec} R \to \operatorname{Spec} R^G$. Explain the example of $\mu_m \curvearrowright \mathbb{C}[x, y]$ acting as in [Ful93, 2.2], and prove that any affine normal toric surface can be obtained via a quotient by a cyclic group, where the quotient map is toric. Provide the example of weighted projective spaces. Define a *resolution of singularities* [Nab, Def. 10.5] and explain the procedure of fan refinement as in [Ful93, Sec. 2.6], interpreting this geometrically as blow-ups. Prove that any proper toric variety can be resolved as explained in [Ful93, Sec. 2.6]. Provide illustrative examples, avoiding any mention of "flops".

Main reference: [Ful93, Sec. 2.2, 2.6]. See also [Nab, Sec. 10].

Talk 10: Divisors on Toric Varieties

Recall the definitions of *Weil divisors*, the *Class Group*, *Cartier divisors*, and the *Picard group*. Recall without proof the inclusion $\operatorname{Pic}(X) \subset \operatorname{Cl}(X)$ for normal varieties and the excision exact sequence for closed subvarieties $Z \subset X$. Define *toric hypersurfaces* and introduce *T-invariant Weil divisors*. Provide the formula for div of a torus character, and finally state and prove the exact sequence in [Nab, Thm. 11.13]. Compute class groups for $\mathbb{P}^n$ and $\mathbb{P}^1 \times \mathbb{P}^1$. Characterize Cartier divisors on normal affine toric varieties, providing an example of a Weil divisor that is not Cartier. Show that every normal affine toric variety has a trivial Picard group. Introduce piecewise linear functions on a fan, and state and prove [Nab, Thm. 11.24]. Finally, state [Nab, Thm. 11.25] and provide the example of a non-factorial variety. Conclude by stating and proving that $X(\Delta)$ is smooth if and only if it is factorial [CLS11, Prop 4.2.6], and that $X(\Delta)$ is $\mathbb{Q}$-factorial if and only if Δ is a simplicial fan [CLS11, Prop. 4.2.7].

Main references: [Nab, Sec. 11, from Def. 11.9], [CLS11, Sec. 4.1, 4.2]. See also [Ful93, Sec. 3.3] and [Tel22, Sec. 5].

Talk 11: Projective Toric Varieties and Polytopes

Define the *associated lattice polytope* P_D of a Cartier toric divisor D on a proper variety, then state and prove the formula for the global sections of $\mathcal{O}(D)$ via P_D. Briefly recall the Kodaira map, the notion of the base locus, and the definitions of ample and very ample divisors. Characterize basepoint-free Cartier toric divisors via the *support function* as in [Nab, Thm. 12.7]. Show that this is equivalent to the convexity of φ_D [CLS11, Thm. 6.1.10]. Provide an example and non-example of convex function in the two-dimensional case (e.g., [CLS11, Ex. 6.1.6]). Define *strict convexity* and prove the equivalence with [CLS11, Lemma 6.1.13(d)]. Characterize the ampleness of a Cartier toric divisor D via φ_D [CLS11, Thm 6.1.14]. Finally, define the *normal fan* of a *lattice polytope* and state [CLS11, Cor. 6.1.15], characterizing projective toric varieties $X(\Delta)$ as those where Δ is the normal fan of some polytope. In particular, note that any two-dimensional normal proper toric variety is projective. Conclude by providing an example of a smooth non-projective toric variety ([CLS11, Ex. 6.1.18]).

Main references: [Nab, Sec. 12], [CLS11, Sec. 6.1]. **Note**: the support function is sometimes defined as $-\varphi_D$; ensure notation remains consistent throughout the seminar.

Talk 12: Toric Varieties as GIT Quotients

Recall the affine GIT quotient construction, defining the G-linearization of line bundles, the notions of semistable and stable points with respect to a line bundle, and the resulting quotient. State the main result: recovering any normal separated toric variety as a quotient, in analogy with the construction $(\mathbb{A}^n \setminus \{0\})/\mathbb{G}_m$:

$$X(\Delta) = \left(\mathbb{A}^{\Delta(1)} \setminus Z(\Delta)\right) /\!/ G.$$

Describe the components of this formula: define $Z(\Delta)$ following [CLS11, Sec. 14.2] and explain the group G as in [CLS11, Ex. 14.1.4]. Prove that $Z(\Delta)$ (and $Z(\Delta')$) can be described as the unstable (non-stable) locus with respect to specific characters of G [CLS11, Prop. 14.1.9]. This completes the proof of [Cox14, Thm. 2.1].

Main reference: [CLS11, Sec. 14.1, 14.2]; see also [Nab, Sec. 13], [Tel22, Sec. 6.1]. This talk is an exposition of a result in [Cox14]. **Note**: For coherence with previous seminars, we use Δ for the fan instead of Σ.

References

[CLS11] David A. Cox, John B. Little, and Henry K. Schenck. *Toric Varieties.* American Mathematical Society, 2011.

[Cox14] David A. Cox. Erratum to "the homogeneous coordinate ring of a toric variety", along with the original paper. https://arxiv.org/abs/alg-geom/9210008, 2014.

[Ful93] William Fulton. *Introduction to Toric Varieties.* Princeton University Press, 1993.

[Nab] Navid Nabijou. Toric geometry. https://drive.google.com/file/d/1tDTVmQxA8mbqf8wxCN-Es7fULmF4A5-x/view. Notes typed by Cat Rust.

[Pega] Christoph Pegel. A crash course on toric varieties, part 1. https://www.iazd.uni-hannover.de/fileadmin/iazd/Pegel/talk-toric1.pdf. Lecture notes.

[Pegb] Christoph Pegel. A crash course on toric varieties, part 2. https://www.iazd.uni-hannover.de/fileadmin/iazd/Pegel/talk-toric2.pdf. Lecture notes.

[Tel22] Simon Telen. Introduction to toric geometry. https://arxiv.org/abs/2203.01690, 2022.