Talk 5: The Cohomology of Curves and Purity

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These are notes for a talk given in the Babyseminar on *the Weil Conjectures* in the Winter Term 2023/24 at the University of Duisburg-Essen. The main references for this talk are [Mil13] and [Mil80]. I would like to thank the organisers Giulio Marazza and Guillermo Gamarra for discussing numerous questions with me.

Last time, we introduced étale cohomology and we computed

$$H^r(U_{\text{ét}}, \mathbb{G}_m) = \begin{cases} k^\times, & r = 0 \\ \operatorname{Pic}(U), & r = 1 \\ 0, & r > 1 \end{cases}$$

for a connected non-singular curve U over an algebraically closed field k. This time, we shall leverage this computation to compute the cohomology of μ_n for curves. Then, we shall see Poincaré duality for curves and finally, we will see purity for étale cohomology, which will allow us to construct a Gysin exact sequence. That is, it will allow us to relate the cohomology of X to the cohomology of a closed subscheme and its open complement.

For the remainder of the talk let k be an algebraically closed field.

1 The cohomology of μ_n

We have the Kummer exact sequence

$$0 \to \mu_n \to \mathbb{G}_m \xrightarrow{\cdot n} \mathbb{G}_m \to 0.$$

Since we already now the cohomology of $\mathbb{G}_m$, we would like to use it to compute the cohomology of μ_n. In order to do this, we have to understand what multiplication by n does on the Picard group of a curve.

For this, we are going to identify the Picard group of a curve U with the group of Weil divisors. That is, we have the sequence

$$K^\times \to \bigoplus_{x \in U} \mathbb{Z} =: \operatorname{Div}(U) \to \operatorname{Pic}(U) \to 0.$$

Here $K = k(U)$ is the field of rational functions on U, and the sum is over all closed points $x \in U$. The map $K^\times \to \operatorname{Div}(U)$ takes a rational function f to its associated divisor $\operatorname{div}(f)$. (These are the principal divisors.) Denote the divisor corresponding with x by $[x]$. Then we can define for any divisor $D = \sum_{x \in U} n_x[x]$ its *degree* as $\sum_{x \in U} n_x$. (The sums are finite and therefore this is well defined.) The divisor $\operatorname{div}(f)$ has degree 0, since a rational function has, with multiplicities, as many zeroes as poles. Thus, the degree factors as a map $\operatorname{Pic}(U) \to \mathbb{Z}$. We denote the kernel of the degree map $\operatorname{Div}(U) \to \mathbb{Z}$ by $\operatorname{Div}^0(U)$. and the quotient $\operatorname{Div}^0(U)/\operatorname{div}(K^\times)$ by $\operatorname{Pic}^0(U)$. This is, in fact, also the kernel of the degree map $\operatorname{Pic}(U) \to \mathbb{Z}$.

Proposition 1.1 ([Mil13, Prop. 14.1]). *Let X be a complete, connected, non-singular curve over k. The sequence*

$$0 \to \operatorname{Pic}^0(X) \to \operatorname{Pic}(X) \to \mathbb{Z} \to 0$$

is exact. For any integer n relatively prime to the characteristic of k, the map

$$\operatorname{Pic}^0(X) \to \operatorname{Pic}^0(X); z \mapsto nz$$

is surjective with kernel equal to a free $\mathbb{Z}/n\mathbb{Z}$-module of rank $2g$, where g is the genus of X.

Proof. The first part of the statement follows immediately from the previous discussion.

The second part of the statement is a bit more involved. Assume first that $k = \mathbb{C}$. Then we can chose a basis $\omega_1, \dots, \omega_g$ for the holomorphic differentials on the Riemann surface $X(\mathbb{C})$, and a basis $\gamma_1, \dots, \gamma_{2g}$ for $H_1(X(\mathbb{C}), \mathbb{Z}) \cong \pi_1(X(\mathbb{C}))^{\text{ab}}$. Let Λ be the subgroup (*not* sub-vector space) of $\mathbb{C}^g$ generated by the vectors consisting of path integrals

$$\left(\int_{\gamma_i} \omega_1, \dots, \int_{\gamma_i} \omega_g \right) \quad \text{for} \quad i = 1, \dots, 2g.$$

For each pair of points $z_0, z_1 \in X(\mathbb{C})$, we choose a path $\gamma(z_0, z_1)$ from z_0 to z_1, and we let

$$I(z_0, z_1) = \left(\int_{\gamma(z_0,z_1)} \omega_1, \dots, \int_{\gamma(z_0,z_1)} \omega_g \right) \in \mathbb{C}^g.$$

Its image in $\mathbb{C}^g/\Lambda$ is independent of the choice of the path $\gamma(z_0, z_1)$. (Integrals over closed loops are invariant under homotopy, and the closed loop you get from composing two paths from z_0 to z_1 will be (up to homotopy) a linear combination of the γ_i.) We can now extend the map $[z_1] - [z_0] \mapsto I(z_0, z_1)$ by linearity to a homomorphism

$$i\colon \operatorname{Div}^0(X) \to \mathbb{C}^g/\Lambda.$$

The famous theorem of Abel and the equally famous Jacobi Inversion Theorem imply that i induces an isomorphism $\operatorname{Pic}^0(X) \to \mathbb{C}^g/\Lambda$.

Now, clearly for any n, the map $\mathbb{C}^g/\Lambda \to \mathbb{C}^g/\Lambda; x \mapsto nx$ is surjective with kernel $\frac{1}{n}\Lambda/\Lambda \cong (\frac{1}{n}\mathbb{Z}/\mathbb{Z})^{2g} \cong (\mathbb{Z}/n\mathbb{Z})^{2g}$, which completes the proof in the case $k = \mathbb{C}$.

For an arbitrary algebraically closed field k, one can do something similar, but that requires the theory of the Jacobian variety. (That is an abelian variety J of dimension g such that $\operatorname{Pic}^0(X) \cong J(k)$. On J, one can perform the proof using the theory of abelian varieties.) ■

With this result in mind, we can compute the cohomology of μ_n.

Proposition 1.2 ([Mil13, Prop. 14.2]). *Let X be a complete, connected, nonsingular curve over k. For any n prime to the characteristic of k, we have*

$$H^0(X, \mu_n) = \mu_n(k), \quad H^1(X, \mu_n) \cong (\mathbb{Z}/n\mathbb{Z})^{2g}, \quad H^2(X, \mu_n) \cong \mathbb{Z}/n\mathbb{Z},$$

and $H^r(X, \mu_n) = 0$ for $r > 2$. Moreover, the isomorphism for $H^2(X, \mu_n) \cong \mathbb{Z}/n\mathbb{Z}$ is canonical.

Proof. We examine the long exact sequence associated with the Kummer exact sequence

$$\cdots \to H^r(X_{\text{ét}}, \mu_n) \to H^r(X_{\text{ét}}, \mathbb{G}_m) \xrightarrow{\cdot n} H^r(X_{\text{ét}}, \mathbb{G}_m) \to \dots.$$

If we fill in the cohomology computation of $H^r(X_{\text{ét}}, \mathbb{G}_m)$ and use the previous proposition, the statement follows.

Note that the isomorphism on $H^2(X, \mu_n)$ is canonical as it is induced by the degree map on $\operatorname{Pic}(X)$ using the Kummer exact sequence. ■

Proposition 1.3 ([Mil13, Prop. 14.3]). *Let U be a nonsingular curve over k. For any n prime to the characteristic of k and closed point $x \in U$, we have*

$$H^2_x(U, \mu_n) \cong \mathbb{Z}/n\mathbb{Z} \quad \textit{and} \quad H^r_x(U, \mu_n) = 0 \textit{ for } r \neq 2.$$

Proof. Let R be the Henselisation of $\mathcal{O}_{U,x}$. By a consequence of excision, we have for $V = \operatorname{Spec} R$ that

$$H^r_x(U, \mu_n) \cong H^r_x(V, \mu_n).$$

We again want to use the Kummer sequence to compute the cohomology. V is again a curve and thus by the above description, we have $H^r(V, \mathbb{G}_m) = 0$ for $r > 0$. Now the exact sequence of the pair $(V, V \setminus x)$

$$\cdots \to H^{r-1}(V, \mathbb{G}_m) \to H^{r-1}(V \setminus x, \mathbb{G}_m) \to H^r_x(V, \mathbb{G}_m) \to H^r(V, \mathbb{G}_m) \to \dots$$

yields the isomorphisms $H^{r-1}(V \setminus x, \mathbb{G}_m) \to H^r_x(\mathbb{G}_m)$ for $r \geq 2$. Now $V \setminus x = \operatorname{Spec} K$ where K is the field of fractions of R, and Lang's Theorem from last time shows that $H^r(K, \mathbb{G}_m) = 0$ for $r \geq 1$; the theorem shows that $H^r(K, \mathbb{G}_m) = 0$ for $r \geq 2$ since the corresponding Galois module K^{sep} is discrete and for this module particularly it is also shown that H^2 vanishes. Furthermore, $H^1(K, \mathbb{G}_m) \cong \operatorname{Pic}(\operatorname{Spec} K) = 0$. We can now combine this with

$$H^1_x(V, \mathbb{G}_m) \cong H^0(K, \mathbb{G}_M)/H^0(V, \mathbb{G}_m) \cong \mathbb{Z}$$

and get the statement using the long exact sequence associated with the Kummer exact sequence. ∎

Remark ([Mil13, Rmk. 14.4]). Let M be a free $\mathbb{Z}/n\mathbb{Z}$-module of rank 1 and let M also denote the constant sheaf on a variety Y defined by M. Then we have

$$H^r(Y_{\text{ét}}, M) = H^r(Y_{\text{ét}}, \mathbb{Z}/n\mathbb{Z}) \otimes M \cong H^r(Y_{\text{ét}}, \mathbb{Z}/n\mathbb{Z}).$$ ⌟

2 Cohomology with compact support

The next big goal of this talk is to prove Poincaré duality for curves. Classically, Poincaré duality is the statement that for an oriented compact (real) manifold M of dimension n, we have a canonical isomorphism $H^k(M) \cong H_{n-k}(M)$.

In algebraic geometry, we don't have homology, but we can work around that by using the universal coefficient theorem, at least if we take our coefficients in a field F, to get $H^k(M) \cong H_{n-k}(M) \cong \operatorname{Hom}(H^{n-k}(M), F)$. In order to prove this, one would like to get rid of the assumption that M is compact. This is possible, by replacing $H^k(M)$ by cohomology with compact support, yielding $H^k_c(M) \cong \operatorname{Hom}(H^{n-k}(M), F)$. This is equivalent in constructing a canonical perfect pairing

$$H^k_c(M) \otimes H^{n-k}(M) \to F,$$

which is essentially what we are going to do in the next section. (The perfect pairing will be the cup product to $H^n_c(M)$, which after choosing an orientation, is canonically isomorphic to F.)

With this in mind, we now want to define cohomology with compact support. Classically for a topological space U, one defines cohomology groups with compact support as

$$H^r(U, \mathbb{Z}) = \underset{\overrightarrow{Z}}{\operatorname{colim}}\, H^r_Z(U, \mathbb{Z})$$

where the filtered colimit runs over the compact subsets of U.

Unfortunately, this definition does not work in the world of algebraic geometry: If one defines $\Gamma_c(U, \mathcal{F}) = \varinjlim_Z \Gamma_Z(U, \mathcal{F})$ for an étale sheaf on $\mathcal{F}$ (the colimit runs over all complete subvarieties of U, and then derives this functor to the right, one does not get a useful theory. For example, if U is affine and one works over an algebraically closed field, then the cohomology with compact support would not be finitely generated over the constant sheaf $\mathbb{Z}/n\mathbb{Z}$. Therefore, we need a different definition.

On a topological space one has for $j\colon U \to X$ a homeomorphism of the topological space U onto an open subset of a locally compact space X, then

$$H^r_c(U, \mathcal{F}) = H^r(X, j_!\mathcal{F}).$$

(Recall that $j_!$ extends $\mathcal{F}$ by zero for an open immersion j.) In particular, when X is compact, then $H^r_c(U, \mathcal{F}) = H^r(X, \mathcal{F})$.

Definition 2.1. For any torsion sheaf $\mathcal{F}$ on a variety U, we define

$$H^r_c(U, \mathcal{F}) = H^r(X, j_!\mathcal{F})$$

where X is any complete variety containing U as a dense open subvariety and j is the inclusion map. ⌟

An open immersion $j\colon U \to X$ from U into a complete variety X such that $j(U)$ is dense in X is called a *completion* or *compactification* of U. Therefore, we call $H^r_c(U, \mathcal{F})$ the *cohomology groups of $\mathcal{F}$ with compact support.* (The name with complete support would be more logical, but probably because the idea was copied from topology, so was the name.)

This definition rases two questions: First, does every variety admit a completion? And second, are the cohomology groups with compact support independent of the completion? The answer to both of these questions is affirmative. The first was shown by Nagata in 1962, and we will deal with the second question next time (as well as more properties of cohomology groups with compact support).

In the case of curves, we can work ourselves around the second issue by using the following construction: Every curve U has a function field K and there is a one-to-one correspondence between function fields and connected, complete, regular curves. Thus, we have a connected, complete regular curve X associated with K, and with this comes a canonical open immersion $j\colon U \to X$. This yields a canonical choice X to compute the cohomology with compact support of U, since we have

$$H^r_c(U, \mathcal{F}) = H^r(X, j_!\mathcal{F}).$$

Because $j_!$ is exact, we can get long exact sequences on cohomology groups from short exact sequences of sheaves on U. Since $j_!$ does not preserve injectives, it is not the rth right derived functor of $H^0_c(U, -)$.

Proposition 2.2 ([Mil13, Prop. 14.5])**.** *For any connected regular curve U over k and integer n not divisible by the characteristic of k, there is a canonical isomorphism*

$$H^2_c(U; \mu_n) \to \mathbb{Z}/n\mathbb{Z}.$$

Proof. We look at the exact sequence associated with the $j_!$ from Talk 4 [Mil13, Prop. 8.15]. So let $j\colon U \to X$ be the canonical completion; in particular j is an open immersion. Let $i\colon Z \to X$ be the closed immersion of the complement. Then, Z is zero-dimensional (and thus a finite collection of $\operatorname{Spec} k$'s). Then, we have the short exact sequence

$$0 \to j_!j^*\mu_n \to \mu_n \to i_*i^*\mu_n \to 0,$$

which in turn yields the long exact sequence on cohomology groups

$$\cdots \to H_c^r(U, \mu_n) \to H^r(X, \mu_n) \to H^r(X, i_* i^* \mu_n) \to \cdots .$$

Now we know

$$H_c^r(X, i_* i^* \mu_n) \cong H^r(Z, i^* \mu_n) = 0$$

for $r > 0$ as Z is zero-dimensional and k is algebraically closed. Therefore, Proposition 1.2 yields

$$H_c^2(U, \mu_n) \cong H^2(X, \mu_n) \cong \mathbb{Z}/n\mathbb{Z},$$

where all isomorphisms are canonical. ■

Remark. For any $x \in U$, sheaf $\mathcal{F}$ on U and $r \geq 0$, there is a canonical map $H_x^r(U, \mathcal{F}) \to H_c^r(U, \mathcal{F})$, as we shall see next time. For $\mathcal{F} = \mu_n$ and $r = 2$, the map is compatible with the isomorphisms in Proposition 1.3 and Proposition 2.2. ⌟

3 Poincaré Duality

Throughout this section U is a connected regular curve over k and n is an integer not divisible by the characteristic of k.

Theorem 3.1 (Poincaré Duality, [Mil13, Thm. 14.7])**.** *For any finite locally constant sheaf $\mathcal{F}$ on U and integer $r \geq 0$, there is a canonical perfect pairing of finite groups*

$$H_c^r(U, \mathcal{F}) \times H^{2-r}(U, \check{\mathcal{F}}(1)) \to H_c^2(U, \mu_n) \cong \mathbb{Z}/n\mathbb{Z}.$$

We have talked about the involved cohomology theories. In order to understand the statement, we still need to talk about three things: We need to define $\check{\mathcal{F}}(1)$, we need to say what it means for a pairing to be perfect, we need to say what we mean by a finite locally constant sheaf, and we need to construct the paring itself.

A pairing $M \times N \to C$ is said to be *perfect* if the induced maps $M \to \mathrm{Hom}(N, C)$ and $N \to \mathrm{Hom}(M, C)$ are isomorphisms. The statement of Poincaré duality thus says that the groups $H^r(U, \mathcal{F})$ and $H^{2-r}(U, \check{\mathcal{F}}(1))$ are dual with respect to this particular pairing.

A sheaf $\mathcal{F}$ on U is said to be *finite locally constant* if it is locally constant, killed by n, and has finite stalks. Thus, for some finite étale covering $U' \to U$, the sheaf $\mathcal{F}|_{U'}$ is the constant sheaf on U' defined by a finite $\mathbb{Z}/n\mathbb{Z}$-module M, and to give a finite locally constant sheaf $\mathcal{F}$ on U is the same as to give a finite $\mathbb{Z}/n\mathbb{Z}$-module endowed with a continuous action of $\pi_1(U, \overline{u})$.

The sheaf $\check{\mathcal{F}}(1)$ is defined to be

$$V \mapsto \mathrm{Hom}_V(\mathcal{F}|_V, \mu_n|_V).$$

Then we have for $\mathcal{G} := \check{\mathcal{F}}(1)$ that $\mathcal{F} = \check{\mathcal{G}}(1)$.

Next, we shall construct the pairing. In order to do that, we need to talk about Ext-groups. For any variety X let $\mathrm{Sh}(X, n)$ be the full subcategory of $\mathrm{Sh}(X_{\text{ét}})$ consisting of sheaves of $\mathbb{Z}/n\mathbb{Z}$-modules. For any étale sheaf on X, we have $H^r(X, \mathcal{F}) = \mathrm{Ext}_X^r(\mathbb{Z}, \mathcal{F})$. If $\mathcal{F}$ happens to be a sheaf of $\mathbb{Z}/n\mathbb{Z}$-modules, we can also compute the cohomology group as $\mathrm{Ext}_{X,n}^r(\mathbb{Z}/n\mathbb{Z}, \mathcal{F})$, where the index X, n means computing the ext group in the category $\mathrm{Sh}(X, n)$.

Now, we can define a pairing (in any abelian category, with objects A, B, C)

$$\mathrm{Ext}^r(A, B) \times \mathrm{Ext}^s(B, C) \to \mathrm{Ext}^{r+s}(A, C)$$

as follows. The elements of $\operatorname{Ext}^r(A,B)$, can be interpreted as equivalence classes of r-fold extensions, i.e. an exact sequence

$$0 \to B \to E_1 \to \cdots \to E_r \to A \to 0.$$

Given an s-fold extension

$$0 \to C \to E'_1 \to \cdots \to E'_s \to B \to 0,$$

we can splice these together to an $(r+s)$-fold extension

$$0 \to C \to E'_1 \to \cdots \to E'_s \to E_1 \to \dots E_r \to A \to 0,$$

which in turn defines an element of $E^{r+s}(A,C)$. This is, in fact, well defined and there are other ways to define the pairing. For example, one could construct the pairing using iterated boundary maps or by using composition in the derived category. Fortunately, all of these definitions yield the same pairing.

In order to use this pairing for Theorem 3.1, we have to work slightly: Let $j\colon U \to X$ be a compactification of our curve. Then, we have the pairing

$$H^r_c(U,\mathcal{F}) \times \operatorname{Ext}^{2-r}_{X,n}(j_!\mathcal{F}, j_!\mu_n) = \operatorname{Ext}^r_{X,n}(\mathbb{Z}/n\mathbb{Z}, j_!\mathcal{F}) \times \operatorname{Ext}^{2-r}_{X,n}(j_!, \mathcal{F}, j_!\mu_n) \to \operatorname{Ext}^2_{X,n}(\mathbb{Z}/n\mathbb{Z}, j_!\mu_n) = H^2_c(U,\mu_n).$$

To identify the last ext group in there, we use that for an extension

$$0 \to \mu_n \to E_1 \to \dots E_{2-r} \to \mathcal{F} \to 0,$$

we can apply the functor $j_!$, which is exact. This yields a map $\operatorname{Ext}^{2-r}_{U,n}(\mathcal{F},\mu_n) \to \operatorname{Ext}^{2-r}_{X,n}(j_!\mathcal{F}, j_!\mu_n)$. Now, we can combine this with the following lemma to obtain the pairing for Poincaré duality.

Lemma 3.2 ([Mil13, Prop. 14.21]). *Let $\mathcal{F}$ be locally constant, then $\operatorname{Ext}^r_{U,n}(\mathcal{F},\mu_n) \cong H^r(U, \check{\mathcal{F}}(1))$.*

Proof. For arbitrary sheaves $\mathcal{F}$ and $\mathcal{G}$, we can define a sheaf-hom $\underline{\operatorname{Hom}}(\mathcal{F},\mathcal{G})$ via $V \mapsto \operatorname{Hom}(\mathcal{F}|_V, \mathcal{G}|_V)$. Now, for $\mathcal{F}_0$ a locally constant sheaf of $\mathbb{Z}/n\mathbb{Z}$ modules with finite stalks, the functor $\underline{\operatorname{Hom}}(\mathcal{F}_0, -)\colon \operatorname{Sh}(U,n) \to \operatorname{Sh}(U,n)$ is left-exact, so we can derive it to the right to get local ext groups $\underline{\operatorname{Ext}}^s$. We can relate these to global ext groups using a Grothendieck spectral sequence – we compose the sheaf-hom with global sections to get the ordinary Hom. That is, we have

$$H^r(U, \underline{\operatorname{Ext}}^s(\mathcal{F}_0,\mathcal{G})) \Rightarrow \operatorname{Ext}^{r+s}(\mathcal{F}_0,\mathcal{G}).$$

Under the given assumptions, one can show that $\underline{\operatorname{Ext}}^s(\mathcal{F}_0,\mathcal{G}) = 0$ for $s > 0$, which yields the statement for the $\mathcal{F}$ from the lemma. ∎

This actually leads to a generalisation of Poincaré duality.

Theorem 3.3 (Poincaré Duality, [Mil13, Thm. 14.20]). *Let U be a nonsingular curve over k. For all constructible sheaves $\mathcal{F}$ of $\mathbb{Z}/n\mathbb{Z}$-modules on U and all $r \geq 0$, there is a canonical perfect pairing of finite groups*

$$H^r_c(U,\mathcal{F}) \times \operatorname{Ext}^{2-r}_{U,n}(\mathcal{F},\mu_n) \to H^r_c(U,\mu_n) \cong \mathbb{Z}/n\mathbb{Z}.$$

Definition 3.4. A sheaf $\mathcal{F}$ on X is called *constructible*, if X can be covered with finitely many locally closed subschemes $Y \subset X$ such that on each Y, the sheaf $\mathcal{F}|_Y$ is finite locally constant. ⌟

Equivalently [Mil80, Ch. 5, Prop 1.8], $\mathcal{F}$ is constructible if for all irreducible, closed subschemes Z of X, there is a non-empty open $U \subset Z$ such that $\mathcal{F}|_U$ is locally constant with finite stalks. In particular, if we choose $Z = X$, we find that there is a non-empty open subset $U \subset X$ such that $\mathcal{F}|_U$ is finite locally constant.

One obtains the more general version of Poincaré duality from Theorem 3.1 using a spectral sequence argument and a reduction to the open subset on which $\mathcal{F}$ is locally constant.

After taking a look at everything that goes into the statement of Poincaré duality for curves, we shall now take a look at the proof of Theorem 3.1. The proof is done in a sequence of steps.

First, we deal with all cases where the pairing (should be) a pairing between zero-groups, that is if $r \neq 0, 1, 2$.

Step 0 For any torsion sheaf $\mathcal{F}$, the group $H^r(U, \mathcal{F})$ and $H^r_c(U, \mathcal{F})$ are zero for $r > 2$.

This is a consequence of a more general vanishing result for étale cohomology, since $H^r_c(U, \mathcal{F}) = H^r(X, j_!\mathcal{F})$.

Theorem 3.5 ([Mil13, Thm. 15.1]). *For any variety X over k, we have*

$$H^r(X, \mathcal{F}) = 0$$

for all torsion sheaves $\mathcal{F}$ on X if $r > 2 \dim X$.

Moreover, if X is affine, then we have the above vanishing if $r > \dim X$.

This proves Theorem 3.1 in all cases $r \neq 0, 1, 2$. For the remaining cases we introduce the following notation: Write $T^r(U, \mathcal{F}) = H^{2-r}_c(U, \check{\mathcal{F}}(1))^\vee$, where $(-)^\vee$ is the dual in the sense of $\mathrm{Hom}(-, \mathbb{Z}/n\mathbb{Z})$. Because in the category of $\mathbb{Z}/n\mathbb{Z}$-modules, a finite module M is canonically isomorphic to $(M^\vee)^\vee$, we only have to prove that the map

$$\varphi^r(U, \mathcal{F})\colon H^r(U, \mathcal{F}) \to T^r(U, \mathcal{F})$$

induced by the pairing is an isomorphism of finite groups for all finite locally constant sheaves $\mathcal{F}$ on U.

Dualising preserves exactness and therefore $\mathcal{F} \mapsto \check{\mathcal{F}}(1)$ preserves exact sequences. So a short exact sequence of finite locally constant sheaves gives rise to a long exact sequence on the $T^r(U, -)$.

Step 1 Let $\pi\colon U' \to U$ be a finite map. Then the theorem holds for $\mathcal{F}$ on U' if and only if it holds for $\pi_*\mathcal{F}$ on U.

Proof. This follows from the fact that π_* is exact and preserves injectives, so $H^r(U, \pi_*\mathcal{F}) = H^r(U', \mathcal{F})$, and a similar statement for T^r. ■

Step 2 Let $V = U \setminus x$ for some point $x \in U$. Then there is an exact commutative diagram with isomorphisms where indicated

$$\begin{array}{ccccccc}
H^r_x(U, \mu_n) & \longrightarrow & H^r(U, \mu_n) & \longrightarrow & H^r(V, \mu_n) & \longrightarrow & H^{r+1}_x(U, \mu_n) \\
\downarrow \cong & & \downarrow \varphi^r(U,\mu_n) & & \downarrow \varphi(V,\mu_n) & & \downarrow \cong \\
H^{2-r}(x, \mathbb{Z}/n\mathbb{Z})^\vee & \longrightarrow & H^{2-r}_c(U, \mathbb{Z}/n\mathbb{Z})^\vee & \longrightarrow & H^{2-r}_c(V, \mathbb{Z}/n\mathbb{Z})^\vee & \longrightarrow & H^{3-r}(x, \mathbb{Z}/n\mathbb{Z})^\vee.
\end{array}$$

(We are interested in this step because this allows us to relate the cohomology of U with the cohomology of the completion X of U.)

Proof. The upper sequence is the long exact sequence of the pair (U, V). The lower sequence is the compact cohomology sequence of

$$0 \to j_! \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to i_* i^* \mathbb{Z}/n\mathbb{Z} \to 0$$

where j and i are the inclusions of V and x into U, respectively.

By Proposition 1.3, we have $H^r_x(V, \mu_n) = H^0(x, \mathbb{Z}/n\mathbb{Z})$ when $r = 2$ and the groups are zero otherwise. The isomorphism for $r = 2$ comes from the pairing of Poincare Duality for the point and one needs to put in some effort to see that the diagram commutes. ■

Step 3 The map $\varphi^0(U, \mathbb{Z}/n\mathbb{Z})$ is an isomorphism of finite groups.

Proof. The pairing from Poincaré Duality

$$H^0(U, \mathbb{Z}/n\mathbb{Z}) \times H^2_c(U, \mu_n) \to H^2_c(U, \mu_n) \cong \mathbb{Z}/n\mathbb{Z}$$

can be identified with the canonical action of $H^0(U, \mathbb{Z}/n\mathbb{Z}) = \mathbb{Z}/n\mathbb{Z}$ on $H^2_c(U, \mu_n)$. ■

Step 4 The theorem is true for $r = 0$ and $\mathcal{F}$ finite locally constant.

Proof. By assumption on $\mathcal{F}$, we find a finite étale covering $U' \to U$ such that $\mathcal{F}|_{U'}$ is constant. Therefore, the idea is now to relate this back to Step 3. We can inject $\mathcal{F}|_{U'}$ into some $\mathcal{F}' = (\mathbb{Z}/n\mathbb{Z})^s$. After applying π_* to the inclusion $\mathcal{F}|_{U'} \to \mathcal{F}'$ and composing the result with the natural inclusion $\mathcal{F} \to \pi_*\pi^*\mathcal{F}$, we obtain the first map in the sequence

$$0 \to \mathcal{F} \to \pi_*\mathcal{F}' \to \mathcal{F}'' \to 0.$$

If we choose $U' \to U$ to be a Galois cover, one can show that $\mathcal{F}''$ is again locally constant. Now we can consider the diagram induced by φ^0 and the associated long exact sequences

$$\begin{array}{ccccccccc}
\bullet & \longrightarrow & \bullet & \longrightarrow & H^0(U, \mathcal{F}) & \longrightarrow & H^0(U, \pi_*\mathcal{F}') & \longrightarrow & H^0(U, \mathcal{F}'') \\
\downarrow\cong & & \downarrow\cong & & \downarrow & & \downarrow\cong & & \downarrow \\
\bullet & \longrightarrow & \bullet & \longrightarrow & T^0(U, \mathcal{F}) & \longrightarrow & T^0(U, \pi_*\mathcal{F}') & \longrightarrow & T^0(U, \mathcal{F}''),
\end{array}$$

where the right hand isomorphism comes from Steps 1 and 3. Now the result follows by applying variations of the five-lemma twice. ■

Step 5 The map $\varphi^1(U, \mu_n)$ is injective.

Proof. We leverage the fact that one can show $H^1(U, \mathbb{Z}/n\mathbb{Z}) = \mathrm{Hom}_{\mathrm{conts}}(\pi_1(U, \bar{u}), \mathbb{Z}/n\mathbb{Z})$. Thus for $s \in H^1(U, \mathbb{Z}/n\mathbb{Z})$, we have a Galois cover $\pi\colon U' \to U$ corresponding with the kernel of s. If we denote the cokernel of $\mathbb{Z}/n\mathbb{Z} \to \pi_*(\mathbb{Z}/n\mathbb{Z})$ by $\mathcal{F}''$, we get the following diagram with exact rows

$$\begin{array}{ccccccc}
H^0(U, \pi_*\mathbb{Z}/n\mathbb{Z}) & \longrightarrow & H^0(U, \mathcal{F}'') & \longrightarrow & H^1(U, \mathbb{Z}/n\mathbb{Z}) & \longrightarrow & H^1(U, \pi_*\mathbb{Z}/n\mathbb{Z}) \\
\downarrow\cong & & \downarrow\cong & & \downarrow & & \downarrow \\
T^0(U, \pi_*\mathbb{Z}/n\mathbb{Z}) & \longrightarrow & T^0(U, \mathcal{F}'') & \longrightarrow & T^1(U, \mathbb{Z}/n\mathbb{Z}) & \longrightarrow & T^1(U, \pi_*\mathbb{Z}/n\mathbb{Z}).
\end{array}$$

(We get the isomorphisms from Step 4.) By construction s maps to zero under the map $H^1(U, \mathbb{Z}/n\mathbb{Z}) \to H^1(U, \pi_*\mathbb{Z}/n\mathbb{Z})$. Now a diagram chase shows that if s is in the kernel of $\varphi^1(U, \mathbb{Z}/n\mathbb{Z})$ then it has to be zero in $H^1(U, \mathbb{Z}/n\mathbb{Z})$. ■

Step 6 The maps $\varphi^r(U, \mathbb{Z}/n\mathbb{Z})$ are isomorphisms of finite groups for $r = 1, 2$.

Proof. We start with $U = X$. Then, we know the involved cohomology groups by earlier computations and we know by Step 6 that the map $\varphi^1(X, \mathbb{Z}/n\mathbb{Z})$ is injective. As both sides have the same rank, $\varphi^1(X, \mathbb{Z}/n\mathbb{Z})$ is an isomorphism. For $r = 2$, we have to show that the pairing

$$H^2(X, \mathbb{Z}/n\mathbb{Z}) \times \mu_n(k) \to H^2(X, \mu_n)$$

is perfect. This follows from the remark after Proposition 1.3.

In order to deduce the statement for U from the one of X, we remove the points of $X \setminus U$ one at a time and use Step 2 together with the five-lemma. ■

Step 7 The maps φ^r are isomorphisms of finite groups for $r = 1, 2$ and $\mathcal{F}$ finite locally constant.

Proof. Repeat what we did in step 4. ■

4 Cohomology purity and the Gysin sequence

Cohomology is contravariant in maps of schemes. Therefore, one can ask when it is also possible to construct maps in the other direction, i.e. given a map of schemes $X \to Y$ when can we construct a canonical map from the cohomology of X to the cohomology of Y. The case when this map is a closed immersion is a consequence of so called cohomology purity, and it shall allow us to compare the cohomology of a scheme X with the cohomology of a closed subscheme and its open complement. This is the Gysin sequence.

Cohomology purity involves a twist of the sheaf that we are taking the cohomology groups of, so we are going to define this first. Fix an integer $n > 0$ and let $\Lambda = \mathbb{Z}/n\mathbb{Z}$. For any ring R such that n is a unit in R, we define $\mu_n(R)$ to be the group of n-th roots of 1 in R, and we define

$$\mu_n(R)^{\otimes r} = \begin{cases} \mu_n(R) \otimes \cdots \otimes \mu_n(R), & r \text{ copies}, r > 0 \\ \Lambda, & r = 0 \\ \mathrm{Hom}(\mu_n(R)^{\otimes -r}, \Lambda), & r < 0. \end{cases}$$

When R is an integral domain and contains all n-th roots of unity, each of the μ_n is a free module of rank 1 over Λ, and the choice of a primitive n-th root of 1 determines a basis for all of them simultaneously.

Let X be a variety over k and assume that $\operatorname{char} k$ does not divide n (alternatively, we can assume that X is a scheme with $n\mathcal{O}_X = \mathcal{O}_X$). We now define $\Lambda(r)$ to be the sheaf on $X_{\text{ét}}$ such that

$$\Gamma(U, \Lambda(r)) = \mu_n(\Gamma(U, \mathcal{O}_U))^{\otimes r}$$

for all $U \to X$ étale and affine. Since k is algebraically closed, it contains all n-th roots of unity and therefore a choice of primitive n-th root of 1 in k determines isomorphisms $\Lambda(r) \to \Lambda$ for all r. In particular, $\Lambda(r)$ constant. For a sheaf $\mathcal{F}$ on $X_{\text{ét}}$ killed by n, we define

$$\mathcal{F}(r) := \mathcal{F} \otimes \Lambda(r)$$

for all $r \in \mathbb{Z}$.

A *smooth pair* (Z, X) of k-varieties is a nonsingular k-variety X together with a nonsingular subvariety Z. We say that (Z, X) has codimension c if every connected component of Z has codimension c in the corresponding component of X.

Theorem 4.1 (Purity, [Mil13, Thm. 16.1])**.** *For any smooth pair of k-varieties (Z, X) of codimension c and locally constant sheaf $\mathcal{F}$ of Λ-modules on X, there are canonical isomorphisms*

$$H^{r-2c}(Z, \mathcal{F}(-c)) \to H^r_Z(X, \mathcal{F})$$

for all $r \geq 0$.

Corollary 4.2 ([Mil13, Cor 16.2])**.** *In the situation of the theorem, there are isomorphisms*

$$H^r(X, \mathcal{F}) \to H^r(U, \mathcal{F})$$

*for $0 \leq r < 2c - 1$ and an exact sequence (*the Gysin sequence*)*

$$0 \longrightarrow H^{2c-1}(X, \mathcal{F}) \longrightarrow H^{2c-1}(U, \mathcal{F}|_U) \longrightarrow H^0(Z, \mathcal{F}(-c)) \longrightarrow H^{2c}(X, \mathcal{F}) \longrightarrow H^{2c}(U, \mathcal{F}|_U) \longrightarrow \cdots \longrightarrow H^{r-2c}(Z, \mathcal{F}(-c)) \longrightarrow H^r(X, \mathcal{F}) \longrightarrow H^r(U, \mathcal{F}) \longrightarrow \cdots$$

Proof. We can look at the long exact sequence for the pair $(X, X \setminus Z)$ and in that use the theorem to replace the groups $H^r_Z(X, \mathcal{F})$ with the groups $H^{r-2c}(Z, \mathcal{F}(-c))$. Since negative cohomology groups are zero, this yields the statement. ■

Example 4.3 (The cohomology of $\mathbb{P}^n$, [Mil13, Example 16.3])**.** The scheme $\mathbb{A}^1$ is a curve and thus, by the very beginning of this talk, we have $H^1(\mathbb{A}^1, \mathbb{G}_m) = \operatorname{Pic}(\mathbb{A}^1) = 0$, since $k[T]$ is a principal ideal domain. Thus, we have $H^r(\mathbb{A}^1, \mathbb{G}_m) = 0$ for all $r > 0$ and the Kummer exact sequence

$$0 \to H^0(\mathbb{A}^1, \mu_n) \to \underbrace{H^0(\mathbb{A}^1, \mathbb{G}_m)}_{=k^\times} \xrightarrow{n} \underbrace{H^0(\mathbb{A}^1, \mathbb{G}_m)}_{=k^\times} \to H^1(\mathbb{A}^1, \mu_n) \to 0$$

yields that $H^r(\mathbb{A}^1, \mu_n) = 0$ for $r > 0$. If we leap forward a few talks, to Talk 7, we can use the Künneth formula to prove that $H^r(\mathbb{A}^m, \Lambda) = 0$ for $r > 0$, that is $\mathbb{A}^m$ is *acyclic*. Therefore, the Gysin sequence for $(\mathbb{P}^{m-1}, \mathbb{P}^m)$ shows that

$$H^0(\mathbb{P}^m, \Lambda) \cong H^0(\mathbb{A}^m, \Lambda) \cong \Lambda, \quad H^1(\mathbb{P}^m, \Lambda) \hookrightarrow H^1(\mathbb{A}^m, \Lambda) = 0 \quad \text{and} \quad H^{r-2}(\mathbb{P}^{m-1}, \Lambda(-1)) \cong H^r(\mathbb{P}^m, \Lambda),$$

for $r \geq 2$. Now, an induction argument shows that

$$H^r(\mathbb{P}^m, \Lambda) = \begin{cases} \Lambda(-\frac{r}{2}), & r \text{ even}, \leq 2m; \\ 0, & \text{otherwise.} \end{cases}$$

(Incidentally, this essentially is the cohomomolgy of complex projective space with Λ-coefficients.) ⌟

Example 4.4 (Cohomology of a smooth hypersurface of $\mathbb{P}^{m+1}$, [Mil13, Ex. 16.4])**.** Let X be a nonsingular hypersurface in $\mathbb{P}^{m+1}$, i.e. a closed subvariety of $\mathbb{P}^{m+1}$ whose homogeneous ideal is $I(X) = (f)$ where f is a homogeneous polynomial in $k[T_0, \ldots, T_{m+1}]$ such that the polynomial

$$f, \frac{\partial f}{\partial T_0}, \ldots, \frac{\partial f}{\partial T_{m+1}}$$

don't have a common zero in $\mathbb{P}^{m+1}$. Then, the complement $U = \mathbb{P}^{m+1} \backslash X$ is affine, which implies $H^r(U, \Lambda) = 0$ for $r > m+1$. Therefore, the Gysin sequence yields maps

$$H^r(X, \Lambda) \to H^{r+2}(\mathbb{P}^2, \Lambda(1))$$

which are isomorphisms for $r > m$ and a surjection for $r = m$. Thus, we have $H^r(X, \Lambda) \cong H^{r+2}(\mathbb{P}^{m+2}, \Lambda(1)) \cong H^{r+2}(\mathbb{P}^m, \Lambda)$ for $r > m$ (by the previous example) and $H^m(X, \Lambda) \cong H^m(\mathbb{P}^2, \Lambda) \oplus H^m(X, \Lambda)'$, where $H^m(X, \Lambda)'$ is the kernel of the map $H^m(X, \Lambda) \to H^{r+2}(\mathbb{P}^2, \Lambda(1))$.

If we now apply the general version of Poincaré duality, which we will see in Talk 8, we obtain that

$$H^*(X, \Lambda) \cong H^*(\mathbb{P}^m, \Lambda) \oplus H^m(X, \Lambda)'.$$

One can generalise this to the case where X is a *smooth complete intersection* in $\mathbb{P}^N$ of dimension m. That is if its homogeneous ideal is generated by by $N - m$ polynomials and if the resulting chain

$$\mathbb{P}^N \supset X_{N-1} \supset \cdots \supset X_m = X$$

with $X_r = H_r \cap X_{r+1}$ where H_r is a hypersurface in $\mathbb{P}^N$ consists only of non-singular schemes. The resulting decomposition of the cohomology groups is the same and it is again deduced by an induction, Poincaré Duality and the Gysin sequence. ⌟

In order to prove Theorem 4.1, we need to generalise the statement a bit, and in order to do that, we have to construct a right-adjoint of i_* for a closed immersion $i\colon Z \to X$. Along the lines of what we have seen in Talk 3, one can see that the functors i^* and i_* define an equivalence of categories between the category of étale sheaves on Z and the category of étale sheaves on X supported on Z. Since by talk 3, i_* preserves injectives, we have $H^r(X, i_*\mathcal{F}) = H^r(Z, \mathcal{F})$ for an étale sheaf $\mathcal{F}$ on Z.

Let $U = X \setminus Z$ be the complement of Z in X and denote the open immersion $U \to X$ by j.

For a sheaf $\mathcal{F}$ on X we define $\mathcal{F}^!$ to be the largest subsheaf of $\mathcal{F}$ with support on Z. Therefore, we have for any étale $\varphi\colon V \to X$ that

$$F^!(V) = \Gamma_{\varphi^{-1}(U)} = \ker(\mathcal{F}(V) \to \mathcal{F}(\varphi^{-1}(U))).$$

One can check that this defines a sheaf on X and that it agrees with $\ker(\mathcal{F} \to j_*j^*\mathcal{F})$.

Now, if $\mathcal{G}$ is a sheaf on X with support on Z, we have that any homomorphism $\alpha\colon \mathcal{G} \to \mathcal{F}$ factors through $\mathcal{F}^!$ and therefore

$$\mathrm{Hom}_X(\mathcal{G}, \mathcal{F}^!) = \mathrm{Hom}_X(\mathcal{G}, \mathcal{F}).$$

We now define $i^!\mathcal{F} := i^*\mathcal{F}^!$ to be the corresponding sheaf on Z. Then we have using the above equivalence

$$\mathrm{Hom}_Z(\mathcal{G}, i^!\mathcal{F}) = \mathrm{Hom}_X(i_*\mathcal{G}, \mathcal{F}).$$

Thus, $i^!$ is the right adjoint of i_*. This implies that $i^!$ is left exact and because the left adjoint i_* is exact, it preserves injectives.

With this, we can state (and prove) the generalisation of Theorem 4.1.

Theorem 4.5 (Cohomology Purity, [Mil13, Thm. 16.7])**.** *Let (Z, X) be a smooth pair of algebraic varieties of codimension c. For any locally constant sheaf of Λ-modules on X, we have $R^{2c}i^!\mathcal{F} \cong (i^*\mathcal{F})(-c)$ and $R^r i^!\mathcal{F} = 0$ for $r \neq 2c$.*

Proof of Theorem 4.1 using Theorem 4.5. The functor $\Gamma_Z(X, -)$ splits as

$$\mathrm{Sh}(X_{\text{ét}}) \xrightarrow{i^!} \mathrm{Sh}(Z_{\text{ét}}) \xrightarrow{\Gamma(Z,-)} \mathrm{Ab}.$$

As $i^!$ preserves injectives, we can use a Grothendieck spectral sequence to compute $H^r_Z(X, \mathcal{F})$, that is, we have

$$E_2^{r,s} = H^r(Z, R^s i^! \mathcal{F}) \Rightarrow H^{r+s}(X, \mathcal{F}).$$

Now, Theorem 4.5 computes the left hand side and shows that the spectral sequence degenerates on the E_2-page. When analysing the degenerated spectral sequence, we get Theorem 4.1. ■

Proof idea for Theorem 4.5. First one constructs the map: Up to application of i^* and i_* to sheaves, roughly the following happens. We consider the composition of sheaves on Z

$$\mathcal{F} = \underline{\mathrm{Hom}}_X(\mathbb{Z}/n\mathbb{Z}, \mathcal{F}) \xrightarrow{i_* R^{2c} i^!} \underline{\mathrm{Hom}}_X(i_* R^{2c} i^!(\mathbb{Z}/n\mathbb{Z}), i_* R^{2c} i^!(\mathcal{F})).$$

Using the adjunction of $\underline{\mathrm{Hom}}$ and $\otimes$, and the adjunction of i^* and i_*, we see that this yields a morphism

$$i^*\mathcal{F} \otimes R^{2c} i^!(\mathbb{Z}/n\mathbb{Z}) \to R^{2c} i^!(\mathcal{F}).$$

Now, one can compute that $R^{2c} i^!(\mathbb{Z}/n\mathbb{Z}) \cong \Lambda(-c)$, which yields the desired twist.

One can check étale-locally that the morphism is an isomorphism. But étale locally, a smooth pair is just a standard smooth pair $(\mathbb{A}^{m-c}, \mathbb{A}^m)$. There, one can show the theorem using induction, starting from the case $c = m = 1$, which is Proposition 1.3. ■

References

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