§ (General) base change

Let's consider a cartesian diagram

$$\begin{tikzcd} X_T \arrow[r, "g'"] \arrow[d, "f'"'] & T \arrow[d, "g"] \\ X \arrow[r, "f"] & S \end{tikzcd} \qquad X_T := X \times_S T$$

and $\mathcal{F}$ an étale sheaf on X.

By applying f_* to the unity $1 \to g'_* g'^*$ we get the map $f_*\mathcal{F} \to f_* g'_* g'^* \mathcal{F} \simeq g_* f'_* g'^* \mathcal{F}$ of sheaves on T. By the universal property of $g^* \dashv g_*$ we get $g^* f_* \mathcal{F} \to f'_* g'^* \mathcal{F}$.

Can we derive this map?

More precisely, can we get a map $g^* R^i f_* \mathcal{F} \to R^i f'_* g'^* \mathcal{F}$ for all i?

We can Take injective resolutions:

$$\mathcal{F} \mapsto I^\cdot$$

$$g'^*(I^\cdot) \mapsto J^\cdot$$ (q-iso of complexes where $J^\cdot$ is a complex of inj sheaves, bounded below)

$g'^* I^\cdot$ is a resolution of $g'^*\mathcal{F}$, because g^* exact, so $J^\cdot$ is as well. We get a map $g'^*\mathcal{F} \to J^\cdot$

Define:

$$\varphi : g^* f_* I^\cdot \to f'_* g'^* I^\cdot \to f'_* J^\cdot$$

A map of complexes induces a map of their cohomology grps.

$$H^i(g^* f_* I^\cdot) = g^* H^i(f_* I^\cdot) = g^* R^i f_* \mathcal{F}.$$

$$H^i(f'_* J^\cdot) = R^i f'_* g'^* \mathcal{F}$$ (because $J^\cdot$ is a resolution of $g_*'\mathcal{F}$)

So we got the map $\boxed{g^* R^i f_* \mathcal{F} \to R^i f'_* g'^* \mathcal{F}}$.

Q: When the base change map is iso?

ex/rmk: It is true for finite morphism

§ Proper base change

Thm [Proper base change theorem]:

$$\begin{tikzcd} X_T \arrow[r, "f'"] \arrow[d, "g'"'] \arrow[dr, phantom, "\lrcorner"] & T \arrow[d, "g"] \\ X \arrow[r, "f"] & S \end{tikzcd}$$

f proper

$\mathcal{F}$ torsion sheaf on X

The base change map $g^* R^i f_* \mathcal{F} \xrightarrow{\sim} R^i f'_* g'^* \mathcal{F}$ is iso

---

This theorem is very strong and very general, but also hard to prove. To make our life easier, we fix some very reasonable supplementary assumption

- $\mathcal{F}$ is a sheaf of $\mathbb{Z}/n\mathbb{Z}$-Modules, with n invertible on S
- $f: X \to S$ is projective (The Chow lemma reduces the non projective case to the projective case).

Remarks:

- Let $f = f_1 \circ f_2 : X \xrightarrow{f_2} X' \xrightarrow{f_1} S$.

  If the theorem holds for f_1 & f_2 $\Rightarrow$ It holds for f.

  If it holds for f & f_2, it holds for f_1 $\forall$ sheaves of the form $f_{2*}\mathcal{G}$.

- If $\mathcal{F}^\bullet$ complex of torsion sheaves bounded below and PBC holds for all $\mathcal{F}^n$, then it holds for $\mathcal{F}^\bullet$, i.e.

$$g^* R f_* \mathcal{F}^\bullet \xrightarrow{\sim} R f'_* g'^* \mathcal{F}^\bullet \quad \text{in } D(T).$$

- If $T = \operatorname{Spec} k$, $k = k^{sep}$, $g = s : \operatorname{Spec} k \to S$ geom. point, the theorem says:

$$(R^i f_* \mathcal{F})_s = H^i(X_s, \mathcal{F}_s),$$

where $X_s := X \times_S \operatorname{Spec} k$ geometric fiber,

$\mathcal{F}_s := \mathcal{F}|_{X_s}$ inverse image.

§ Proof of PBC

In this section we'll see a proof of PBC in our setting, or at least a part of it.

- We can assume T finitely generated S-scheme

[Assume T, S affine. T can be written as a ~~the~~ limit of finite S-schemes. The general case follows.]

- We are comparing sheaves on T. A hom. of sheaves on T is iso iff it's iso on all stalks. For finitely generated schemes, points that are closed in their fiber are dense, so we only need to check the ~~statement~~ the iso on the stalks at t_0, with $t_0 \in T$ closed in its fiber.

- Stalks: t spec $\kappa \to T$ geometric pt

$T(t) = \operatorname{Spec} \tilde{\mathcal{O}}_{T,t}$, $\tilde{\mathcal{O}}_{T,t}$ strict local ring

(it's the Henselization of $\mathcal{O}_{T,t}$, or equivalently, ~~the étale st~~ $\varinjlim \operatorname{colim}_{U \text{ étale neigh of } t} \mathcal{O}_T(U)$)

$\mathcal{G}$ sheaf on X_T.

$\mathcal{G}(t) :=$ inverse image on $X_T \times T(t)$.

As a corollary of the compatibility of cohomology with colimits, one can verify:

$(R^i f'_* \mathcal{G})_t \simeq H^i(X_T \times_T T(t), \mathcal{G}(t))$

Similarly, for $s: \operatorname{spec} \kappa \to T$, ~~we have~~ and $\mathcal{F}$ sheaf on X,

~~$R^i f_*$~~ we have: $(R^i f_* \mathcal{F})_s \simeq H^i(X \times_S S(s), \mathcal{F}(s))$.

We want to compare those stalks, when $t = g^{-1}(s)$, and $\mathcal{G} = g^* \mathcal{F}$.

- We can reduce to one of the following two cases:

1) $g: \underset{T}{\underset{\shortparallel}{\operatorname{spec} K}} \to \underset{S}{\underset{\shortparallel}{\operatorname{spec} \kappa}}$, where K finite ~~extension~~ algebraic extension of a separably closed field κ.

2) $S = \operatorname{spec} A$ with A strictly Henselian and $T = \operatorname{spec} \kappa$, with $\kappa = A/\mathfrak{m}$ residue field. $g: T \to S$ natural mapping

Let's explain why:

In case 2), the theorem can be rephrased as: $\circledast$

$S = \operatorname{Spec} A$, with A i strictly Henselian ring, $g = s : \operatorname{Spec} k \to A$, $k = A/m$.

Then $X_s := X \times \operatorname{Spec} k$, & $F_s = F|_{X_s}$, then $H^i(X, F) = H^i(X_s, F_s)$.

If we assume this time, we can apply this formula to the expressions of the stalk at s $\quad H^i(X \times_S S(s), F(s))$

with respect to the situation

$$\begin{tikzcd} & \operatorname{Spec} k \arrow[d] \\ X \times_S S(s) \arrow[r] & S(s) \end{tikzcd}$$

and obtain:

$$H^i(X \times_S S(s), F(s)) = H^i(X_k, F_k)$$

$$(X \times_S S(s)) \times_{S(s)} \operatorname{Spec} k =$$

$$= X \times_S (S(s) \times_{S(s)} \operatorname{Spec} k) = X \times_S \operatorname{Spec} k =$$

$$=: X_k$$

Similarly

$H^i(X_T \times_T T(t), G(t)) = H^i(X_{T,k'}, G_{k'})$, $\quad X_{T,k'} := X_T \times_T \operatorname{Spec} k'$

k' residue field of $\tilde{\mathcal{O}}_{T,t}$.

But the residue fields of a strictly Henselian ring is separably closed, and k' is a finite algebraic extension of k, then, the iso $H^i(X_k, F_k) \simeq H^i(X_{T,k'}, G_{k'})$ follows by the case (2).

(*) This comes from the fact that $T = \operatorname{spec} k$ with $k = k^{sep}$, and in this case

one has the ~~identification~~ that

$R^i f_* \mathcal{G}$ corresponds to the group

$H^i(X_T, \mathcal{G})$ through the equivalence

of categories $\mathcal{F} \mapsto \mathcal{F}(T)$

· Case 1) $T = \mathrm{Spec}\, K \to \mathrm{Spec}\, k$, K finite algebraic extension of a separably closed field k.

In particular, K is a purely inseparable extension of k.

In this case, PBC is a well known property of étale cohomology, that comes as a consequence of compatibility of étale cohomology with projective limits, and the "permanence property":

If $f: X \to Y$ purely inseparable, and G sheaf of abelian grps on Y, then $H^i(Y, G) \simeq H^i(X, f^*G)$.

· Case 2) Now, we are in the situation:

$S = \mathrm{Spec}\, A$, A strictly Henselian ring, $k = A/\mathfrak{m}$ residue field, $g = s: \mathrm{Spec}\, k \to S$. $X_s := X \times \mathrm{Spec}\, k$, $\mathcal{F}_s = \mathcal{F}|_{X_s}$, and we want to prove: $\boxed{H^i(X, \mathcal{F}) = H^i(X_s, \mathcal{F}_s)}$

Firstly, we study the special case where X_s is at most one-dimensional.

We state a preliminary result:

Fact: Assume $\dim X_s \le 1$, n invertible in $\mathcal{O}_X$.
Then, the natural mapping $H^i(X, \mathbb{Z}/n) \to H^i(X_s, \mathbb{Z}/n)$ is bijective for $i=0$, and surjective for $i>0$.

[Since $\dim X_s \le 1$, $H^i(X_s, \mathcal{F}) = 0$ $\forall i > 2$, $\forall \mathcal{F}$, so the statement is trivial for $i>2$.
The 3 cases $i=0,1,2$ have to be handled separately.
They are non-trivial result, widely studied in the Holy Scriptures.]

• Now let's suppose to have $p: X' \to X$ finite morphism, then, the fact holds for X'.

We also know that the base change is iso for finite morphisms, then, if $\mathcal{F}$ is a sheaf on X isomorphic to a finite direct sum of sheaves of the form $P_*(\mathbb{Z}/n\mathbb{Z})$, we have that $H^i(X, \mathcal{F}) \to H^i(X_s, \mathcal{F}_s)$ is bijective for $i=0$
surjective $i>0$.

~~Moreover~~ "Every constructible sheaf (of ($\mathbb{Z}/n\mathbb{Z}$-mod)) is a subsheaf of a constructible sheaf $\mathcal{G}$ with that property ($H^i(X, \mathcal{G}) \to H^i(X_s, \mathcal{G}_s)$ is bijective for $i=0$, surj. for $i>0$)"

Then take a constructible sheaf (of $\mathbb{Z}/n\mathbb{Z}$-mod) $\mathcal{F} \hookrightarrow \mathcal{G}$.

We have

$$\begin{tikzcd}
0 \arrow[r] & H^0(X, \mathcal{F}) \arrow[r] \arrow[d] & H^0(X, \mathcal{G}) \arrow[r] \arrow[d, "\wr\!\!\parallel"] & H^0(X, \mathcal{G}/\mathcal{F}) \arrow[d] \\
0 \arrow[r] & H^0(X_s, \mathcal{F}_s) \arrow[r] & H_0(X_s, \mathcal{G}_s) \arrow[r] & H_0(X_s, (\mathcal{G}/\mathcal{F})_s)
\end{tikzcd}$$

From which we see that $H^0(X, \mathcal{F}) \to H^0(X_s, \mathcal{F}_s)$ is injective. this holds for all constructible sheaves of $\mathbb{Z}/n\mathbb{Z}$-mod, then also for all sheaves of $\mathbb{Z}/n\mathbb{Z}$-mod (as they are built out of constructible sheaves) $\rightsquigarrow$ also for $\mathcal{G}/\mathcal{F}$.

Using that $H^0(X, \mathcal{G}/\mathcal{F}) \to H^0(X_s, (\mathcal{G}/\mathcal{F})_s)$ injective, we see from the diagram that $H^0(X, \mathcal{F}) \to H^0(X_s, \mathcal{F}_s)$ has to be bijective

- Let's finish the $(\dim \leq 1)$-case by proving that $H^i(X, \mathcal{F}) \to H^i(X_s, \mathcal{F}_s)$ is iso by induction on i.

We already have the first step, then let's do the induction:
Suppose $H^i(X, \mathcal{F}) \xrightarrow{\sim} H^i(X_s, \mathcal{F}_s)$ $\forall\, i < p$. Let's prove $H^p(X, \mathcal{F}) \xrightarrow{\sim} H^p(X_s, \mathcal{F}_s)$.

We $\overbrace{\text{can suppose}}^{\to \text{ not even necessary}}$ $\mathcal{F}$ constructible. $\mathcal{F} \hookrightarrow \mathcal{G}$ with $\mathcal{G}$ satisfying the previous property. In particular $H^p(X, \mathcal{G}) \to H^p(X_s, \mathcal{G}_s)$ is surjective ~~injective~~. Let's prove that it is also injective, and hence an iso.

∃ ses of $\mathbb{Z}/m\mathbb{Z}$-modules $0 \to \mathcal{G} \to \mathcal{I} \to \mathcal{K} \to 0$ with $\mathcal{I}$ injective (because $\mathbb{Z}/m\mathbb{Z}$-mod has enough injectives). By applying long exact sequence in cohomology, and using the fact that $\mathcal{I}$ is injective, we can construct a diagram

$$\begin{tikzcd}
H^{p-1}(\mathcal{I}) \arrow[r] \arrow[d, "\simeq"'] & H^{p-1}(\mathcal{K}) \arrow[r] \arrow[d, "\simeq"'] & H^p(\mathcal{G}) \arrow[r] \arrow[d] & 0 \arrow[d] \\
H^{p-1}(\mathcal{I}_\eta) \arrow[r] & H^{p-1}(\mathcal{K}_\eta) \arrow[r] & H^p(\mathcal{G}_\eta) \arrow[r] & H^p(\mathcal{I}_\eta)
\end{tikzcd}$$

First two vertical arrow are iso by induction hp

~~the~~ $\Rightarrow H^p(\mathcal{G}) \to H^p(\mathcal{G}_\eta)$ is injective $\Rightarrow$ it is an iso ✓

For the analogous statement for $\mathcal{F}$, let's consider the diagram

$$\begin{tikzcd}
H^{p-1}(\mathcal{G}) \arrow[r] \arrow[d, "\simeq", "\text{induction}"'] & H^{p-1}(\mathcal{G}/\mathcal{F}) \arrow[r] \arrow[d, "\simeq"] & H^p(\mathcal{F}) \arrow[r] \arrow[d] & H^p(\mathcal{G}) \arrow[r] \arrow[d, "\simeq \text{ previous claim}"] & H^p(\mathcal{G}/\mathcal{F}) \arrow[r] \arrow[d] & \dots \\
H^{p-1}(\mathcal{G}_\eta) \arrow[r] & H^{p-1}((\mathcal{G}/\mathcal{F})_\eta) \arrow[r] & H^{p}(\mathcal{F}_\eta) \arrow[r] & H^p(\mathcal{G}_\eta) \arrow[r] & H^p((\mathcal{G}/\mathcal{F})_\eta) \arrow[r] & \dots
\end{tikzcd}$$

$\Rightarrow H^p(\mathcal{F}) \to H^p(\mathcal{F}_\eta)$ is injective $\xRightarrow{\mathcal{F} \text{ was generic}}$ $H^p(\mathcal{G}/\mathcal{F}) \to H^p((\mathcal{G}/\mathcal{F})_\eta)$ is injective $\Rightarrow H^p(\mathcal{F}) \to H^p(\mathcal{F}_\eta)$ is also surjective.

We conclude $H^p(\mathcal{F}) \to H^p(\mathcal{F}_\eta)$ is iso.

This conclude the thesis for $f: X \to S$ such that $\dim X_\eta \leq 1$.

Now we handle the case with general fibers, starting by:

· <u>PBC for $\mathbb{P}^n_S$</u> (i.e. $X = \mathbb{P}^n_S \to S$).

We know that PBC holds for maps f with at most 1-dim fibers, then, by considering fiber products, it holds for maps that can be written as products of maps of the previous type. In particular, we have PBC for $\mathbb{P}^1_S \times \dots \times \mathbb{P}^1_S \xrightarrow{f} S$.

$\exists$ cover $\mathbb{P}^1_S \times \dots \times \mathbb{P}^1_S \xrightarrow{p} \mathbb{P}^n_S$ (it's the quotient map on $\mathbb{P}^1_S \times \dots \mathbb{P}^1_S / S_n$ sym grp)

PBC holds for sheaves $\mathcal{F}$ on $\mathbb{P}^n_S$ of the form $p_* \mathcal{F}'$, as a consequence of BC for finite morphisms. Moreover, the map $\mathcal{G} \to p_* p^* \mathcal{G}$ is injective for all sheaves $\mathcal{G}$ (as $1 \to p_* p^*$ is the unity map). We can then construct an injective resolution of $\mathcal{G}$ of sheaves of that form:

$$\begin{tikzcd}
\mathcal{G} \arrow[r, hook] & p_* p^* \mathcal{G} \arrow[r, two heads, "Q"] \arrow[d, equal] & p_* p^* \mathcal{G}/\mathcal{G} =: \mathcal{I}^1 \arrow[d, hook] & & \\
& \mathcal{I}^0 & p_* p^* \mathcal{I}^1 \arrow[r, two heads] \arrow[d, equal] & p_* p^* \mathcal{I}^1/\mathcal{I}^0 \arrow[d, hook] & \\
& & \mathcal{G}^1 & p_* p^* \mathcal{I}^2 \arrow[r] \arrow[d, equal] & \dots \\
& & & \mathcal{G}^2 &
\end{tikzcd}$$

$\mathcal{G}^0 \to \mathcal{G}^1 \to \mathcal{G}^2 \to \dots$ is exact

$\mathcal{G}$ (as a complex concentrated in degree 0) is q.-iso to $\mathcal{G}^\bullet$, and PBC holds for all $\mathcal{G}^n$.

Then, from one of the remarks at the beginning, PBC holds for $\mathcal{G}$ (seen as a complex concentrated in degree 0), but it's the same as saying that it holds for $\mathcal{G}$ (seen as a single sheaf)

This finishes the case of $\mathbb{P}^n_S$.

· <u>PBC</u> <u>for projective morphisms</u>:

$f: X \to S$ projective $\Rightarrow$ it factorizes as:

$$\begin{tikzcd} f: X \arrow[rr] \arrow[dr, hook] & & S \\ & \mathbb{P}^n_S \arrow[ur] & \end{tikzcd}$$

(this is valid for S affine, but it's our case: $S = \operatorname{Spec} A$)

The embedding $X \hookrightarrow \mathbb{P}^n_S$ is a finite morphism, thus, again by the initial remark, PBC holds for f. $\square$

B) Cohomology with compact support

As in topology, we want define cohomology with compact support for étale cohomology, in order to have some kind of duality that makes étale cohomology a good cohomology theory (cohomology grp with compact support are usually supposed to provide to ordinary cohomology grps).

<u>First attempt</u>: By analogy with the topological case we could define cohomology with compact support as the derived functor of the functor $\Gamma_c(U, \mathcal{F}) := \varinjlim_{\substack{Z \subseteq U \\ \text{complete}}} \Gamma_Z(U, \mathcal{F})$,

where $\Gamma_Z(-, \mathcal{F})$ is the ~~su~~ functor of sections of $\mathcal{F}$ with support in Z.

This approach fails because it leads to anomolous ~~grp~~ groups.

<u>Exmp</u>: U affine variety $/k=\bar{k}$, Complete subvarieties of

U are finite subvarieties. If Z finite subvariety,

$H^n_Z(U, \mathcal{F}) = \bigoplus_{z \in Z} H^n_z(U, \mathcal{F})$.

If U is smooth of dim m, $\Lambda = \mathrm{Const}_{\mathbb{Z}/n\mathbb{Z}}$, we have

$$H^n_c(U, \Lambda) = \varinjlim H^n_Z(U, \Lambda) = \bigoplus_{z \in U} H^n_z(U, \Lambda) = \begin{cases} \bigoplus_{z \in U} \Lambda(-m) & \text{if } n = 2m \\ 0 & \text{otherwise} \end{cases}$$

this groups are not even finite.

Let's define cohomology with compact support by using our definition of $j_!$

<u>Def</u>: $j : U \hookrightarrow X$ open immersion of U into a complete variety X, with $j(U)$ dense in X. j is called a <u>completion</u> (or <u>compactification</u>) of U

<u>Def</u>: $\mathcal{F}$ torsion sheaf on a variety U. We define the <u>cohomology with compact support</u> of $\mathcal{F}$ through

$H^n_c(U, \mathcal{F}) := H^n(X, j_! \mathcal{F})$ for $j : U \to X$ a compactification.

This is a good definition only if every U admits a compactification and the cohomology groups are independent on the single compactification.

The first issue is non-trivial, but it's a well known fact.

Thm (Nagata, 1962): $\pi\ U \to S$ separated of finite type, with U noetherian $\Rightarrow \exists$ proper morphism $\bar{\pi} : X \to S$ and an open ~~immersion~~ $j : U \hookrightarrow X$ such that $\pi = \bar{\pi} \circ j$.

This solves our first problem

Now we have to check the good definition:

<u>Prop</u>: $H^r(X, j, \mathcal{F})$ is independent of the choice of $j : U \to X$ (i.e. it depends only on U) ($\mathcal{F}$ torsion)

<u>Pf</u>: $j_1 : U \hookrightarrow X_1$, $j_2 : U \hookrightarrow X_2$ two compactifications

This gives $j : U \to X_1 \times X_2$ diagonal, let $X := \overline{j(U)}$

Then, j is still a completion, $X \to X_1$ and $X \to X_2$ are proper. * It's enough to show $H^r(X_1, j_1, \mathcal{F}) \simeq H^r(X, j, \mathcal{F})$.

Consider the diagram
$$\begin{tikzcd} U \arrow[r, hook, "j"] \arrow[d, equal] & X \arrow[d, "\pi"] \\ U \arrow[r, hook, "j_1"] & X_1 \end{tikzcd}$$

We defined the Leray spectral sequence associated to a map

· Consider then the Leray spectral sequence associated to π : ~~$H^r(X_1, R^s\pi_* \mathcal{G}) \Rightarrow H^{r+s}(X, \mathcal{G})$~~

$$H^r(X_1, R^s\pi_* \mathcal{G}) \Rightarrow H^{r+s}(X, \mathcal{G}).$$

Apply this to $\mathcal{G} = j_! \mathcal{F}$. obtain:

$$H^r(X_1, R^s\pi_*(j_! \mathcal{F})) \Rightarrow H^{r+s}(X, j_! \mathcal{F}).$$

· We want to show that $R^s\pi_*(j_! \mathcal{F}) = 0 \quad \forall\, s > 0$.

It's enough to show it is 0 on the ~~geometric points~~ stalks of the geometric points.

$R^s\pi_*(j_! \mathcal{F})$ is a sheaf on X_1, so, take a geometric point $\operatorname{Spec} \Omega \xrightarrow{\bar{x}} X_1$, $\underbrace{}_{\text{with } \Omega = \Omega^{sep}}$ and let's compute the stalk

$(R^s\pi_*(j_! \mathcal{F}))_{\bar{x}}$. This is an application of PBC:

Apply proper base change to:

$$\begin{tikzcd} X_{\bar{x}} \arrow[r] \arrow[d] & \operatorname{Spec} \Omega \arrow[d, "\bar{x}"] \\ X \arrow[r, "\pi"] & X_1 \end{tikzcd}$$

situation with $T = \operatorname{Spec}$ of a separably closed field

By proper base change: $(R^s\pi_*(j_! \mathcal{F}))_{\bar{x}} = H^s(X_{\bar{x}}, (j_! \mathcal{F})|_{X_{\bar{x}}})$

There are two possibilities: $\bar{x}(\operatorname{Spec} \Omega) =: x \in U$,

$x \notin U$

If $x \notin U$, $(j_! \mathcal{F})|_{X_{\bar{x}}} = 0 \Rightarrow \left(R^s \pi_* (j_! \mathcal{F})\right)_{\bar{x}} = 0$ ~~$\forall s \geq$~~ $\forall s \geq 0$.

If $\bar{x} \in U$, the fiber of x is just one point:

$\pi|_U = id_U$, so, if $x \in U$, π is the identity map in an open neighbourhood of x

$$\begin{tikzcd} U \arrow[r, equal] \arrow[d, hook] & U \arrow[d, hook] & \\ X \arrow[r, "\pi"] & X_1 & \operatorname{Spec} \Omega \arrow[l, hook'] \end{tikzcd}$$

then

$X_{\bar{x}} = \operatorname{Spec} \Omega = \bar{x}$. $\Rightarrow H^s(X_{\bar{x}}, (j_! \mathcal{F})|_{X_{\bar{x}}}) = 0$ ~~by dimension~~ $\forall s > 0$

by the dimension. We have then proved

that $R^s \pi_* (j_! \mathcal{F}) = 0 \quad \forall s > 0$

- Since $R^s \pi_* (j_! \mathcal{F}) = 0 \ \forall s > 0$, the Leray spectral sequence degenerates at the second page $\Rightarrow$

$$H^n(X_1, \pi_* (j_! \mathcal{F})) \simeq H^n(X, j_! \mathcal{F}).$$

- Moreover, it's not difficult to prove, but a bit long, that $\pi_* j_! \mathcal{F} = j_{1!} \mathcal{F}$.

We can ~~coan~~ check this on the stalks:

$$(j_{1!}\mathcal{F})_{\bar{x}} = \begin{cases} 0 & \text{if } x \notin U \\ \mathcal{F}_{\bar{x}} & \text{if } x \in U \end{cases}$$

$$(\pi_*(j_!\mathcal{F}))_{\bar{x}} = \begin{cases} 0 & \text{if } x \notin U \\ \mathcal{F}_{\bar{x}} & \text{if } x \in U \end{cases} \quad \text{(because of the def of } j_! \text{ and pullbacks preserve stalks)}$$

This is actually not a proof, since two sheaves with the same stalks are not necessarily isomorphic. Let's ~~check~~ see the sections:

$$\begin{array}{c} V \\ \varphi \downarrow \ \text{étale} \\ X_1 \end{array} \qquad \begin{tikzcd} V\times_{X_1} X \arrow[r] \arrow[d, "\varphi'"'] & V \arrow[d, "\varphi"] \\ X \arrow[r, "\pi"] & X_1 \end{tikzcd}$$

$$(j_{1!}\mathcal{F})(V) = \begin{cases} \mathcal{F}(V) & \text{if } \varphi(V) \subseteq U \\ 0 & \text{otherwise} \end{cases}$$

$$(\pi_* j_! \mathcal{F})(V) = j_!\mathcal{F}(V\times_{X_1} X) = \begin{cases} \mathcal{F}(V\times_{X_1} X) & \text{if } \varphi'(V\times_{X_1} X) \subseteq U \\ 0 & \text{otherwise} \end{cases}$$

By verifying the universal property of the fibre product, we can show that $V \simeq V\times_{X_1} X$, then the two sheaves have the same sections. Then they are isomorphic.

We finally conclude:

$$H^n(X_1, j_{1!}\mathcal{F}) \simeq H^n(X_1, \pi_* j_! \mathcal{F}) \simeq H^n(X, j_!\mathcal{F})$$

$\square$

Prop: 1) $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ s.e.s of sheaves, then it gives rise to a l.e.s in cohomology with compact support.

~~a)~~ 2) If $\mathcal{F}$ constructible $\Rightarrow H^n_c(U, \mathcal{F})$ finite.

Pf: 1) ~~a~~ Let $j: U \to X$ compactification. We know $j_!$ is exact.

$\Rightarrow 0 \to j_! \mathcal{F}' \to$ ~~$\mathcal{F}$~~ $j_! \mathcal{F} \to j_! \mathcal{F}'' \to 0$ is exact $\Rightarrow$ it gives rise long exact sequence in cohomology.

2) $\mathcal{F}$ constructible $\Rightarrow j_! \mathcal{F}$ constructible

~~Thm~~. For a constructible sheaf $\mathcal{G}$ on a complete variety X, $H^n(X, \mathcal{G})$ is finite:

Consider $\pi: X \to s$ map from X to a point.

$\exists$ equivalence of categories $Sh(S_{et})^{constr} \to Ab^{fin}$

$\mathcal{F} \mapsto \mathcal{F}(s)$

Since π proper $R^n \pi_* \mathcal{F} \overset{PBC}{\mapsto} H^n(X, \mathcal{F})$

~~The finiteness is a consequence of the PBC theorem~~

~~Thm Proper.~~

$\square$

Rmk: $H^r_c(U,-)$ is not the rth right derived functor of $H^0_c(U,-)$. $H^r(U,-)$ is not a classical construction in this sense. Also, although $H^r_c(U,-)$ doesn't depend of the choice of completion, there seems to be no good definition of it only in terms of U.

§ Higher direct images with proper support

Let $\pi : U \to S$ be regular map of varieties (it is ok to assume: π separated of finite type).

Even if we didn't define $\pi_!$, we want to define $R^r\pi_!$.

Let $\bar\pi : X \to S$ proper such that $\exists$ $j : U \hookrightarrow X$ open immersion such that

$$\begin{tikzcd} U \arrow[rr, hook, "j"] \arrow[dr, "\pi"'] & & X \arrow[dl, "\bar\pi"] \\ & S & \end{tikzcd}$$

commutes, and $j(U)$ dense in X.

Def: $\forall$ $\mathcal{F}$ torsion sheaf on U, define:

$$R^r\pi_!\mathcal{F} := R^r\bar\pi_*(j_!\mathcal{F}).$$

[I think: for $r=0$ we obtain the "right" definition of $f_!$, that is, the definition of $f_!$ for all f separated of finite type, that generalizes the case of open immersion, and includes the case of proper morphism. There is a natural transformation $f_! \to f_*$, that is invertible if f proper.]

The following properties are easy to verify, with the information coming from the previous section

Prop. 1) The sheaf $R^n\pi_! \mathcal{F}$ is independent of the choise of the factorization $\pi = \bar{\pi} \circ j$

2) A s.e.s. of sheaves gives rise to a l.e.s of higher direct images with proper support.

3) If $\mathcal{F}$ constructible, $R^n\pi_! \mathcal{F}$ is constructible.

Rmk: $R^n\pi_!$ is not the n-th right derived functor of $R^0\pi_!$. (Indeed, $\pi_!$ is an exact functor)

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§ ~~[illegible]~~ Sheaves of $\mathbb{Z}_\ell$-modules

Motivation: We worked so far ~~[illegible]~~ only with torsion sheaf. Indeed Étale cohomology, as we have defined, does not ~~[illegible]~~ well behave for non-torsion sheaves: ~~for instance, if X~~

e.g. X normal: $H^1(X,\mathbb{Z}) = \mathrm{Hom}_{\text{cont.}}(\pi_1(X,\bar{x}), \mathbb{Z}_{\text{discr}}) = 0$

For the same reason $H^1(X,\mathbb{Z}_\ell) = 0$.

But we still need to have cohomology grp ~~[illegible]~~ that are vector spaces over a characteristic 0 field. For instance, because we need a Lefschetz fixed-point formula. Thus:

Problem: our definition works for $H^n(X, \mathbb{Z}/m\mathbb{Z})$ but not for $H^n(X, \mathbb{Z}_\ell)$. We want a replacement for the latter

Def: $H^n(X, \mathbb{Z}_\ell) = \varprojlim_m H^n(X, \mathbb{Z}/\ell^m\mathbb{Z})$

We have to define this way because cohomology does not commute with limits.

With this definition, we have:

$$H^1(X, \mathbb{Z}_\ell) \overset{\text{def}}{=} \varprojlim_m H^1(X, \mathbb{Z}/\ell^m\mathbb{Z}) = \varprojlim_m \mathrm{Hom}_{\mathrm{cont}}(\pi_1(X,\bar{x}), \mathbb{Z}/\ell^m\mathbb{Z})$$

$$\simeq \mathrm{Hom}_{\mathrm{cont}}(\pi_1(X,\bar{x}), \underbrace{(\mathbb{Z}_\ell)_\ell}_{\text{with } \ell\text{-adic}})$$

The last iso can be checked directly.

In order to define sheaves of $\mathbb{Z}_\ell$-module, let's give a characterization of those:

~~A finitely gen.~~

$$\{\text{finitely gen. } \mathbb{Z}_\ell\text{-modules}\} \longleftrightarrow \left\{ \begin{array}{l} \{M_m\}_m,\ M_m \text{ fin. gen. } \mathbb{Z}/\ell^m\mathbb{Z}\text{-mod} \\ \forall m,\ f_m : M_m \to M_{m-1} \text{ such that} \\ \text{the induced map } M_m/\ell^{m-1}M_m \to M_{m-1} \\ \text{is iso} \end{array} \right\}$$

- $M \mapsto \begin{cases} M_m = M/\ell^m M \\ f_{m+1} \text{ quotient map} \end{cases}$

- $\{M_m, f_{m+1}\}_m \mapsto M := \varprojlim_m M_m$.

This is actually an equivalence of categories.

<u>Fact</u>: $M_{m+s}/\ell^m M_{m+s} \simeq M_m$

$\mapsto$ Induction on s:

- $s=0 \rightsquigarrow$ definition
- inductive step: $M_{m+s+1}/\ell^{m+s} M_{m+s+1} \simeq M_{m+s}$ by construction,

taking ℓ^m torsion, we obtain:

$M_{m+s+1}/\ell^m M_{m+s+1} \simeq M_{m+s}/\ell^m M_{m+s} \simeq M_m$ by induction hp $\leftarrow\!|$

Take the s.e.s $\quad 0 \to \mathbb{Z}/\ell^s\mathbb{Z} \xrightarrow{\cdot \ell^m} \mathbb{Z}/\ell^{m+s}\mathbb{Z} \to \mathbb{Z}/\ell^m\mathbb{Z} \to 0$

and tensor ~~once~~ with M_{m+s} over $\mathbb{Z}/\ell^{m+s}\mathbb{Z}$, obtain

$0 \to M_s \to M_{m+s} \to M_m \to 0$ (because

$$M_{m+s} \otimes_{\mathbb{Z}/\ell^{m+s}\mathbb{Z}} \mathbb{Z}/\ell^m\mathbb{Z} \simeq M_{m+s}/\ell^m M_{m+s} \simeq M_m).$$

This sequence is exact if M torsion free (because "torsion free" $\Leftrightarrow$ "flat").

By analogy with this characterization of $\mathbb{Z}_\ell$-modules, we give the following:

Def. A sheaf of $\mathbb{Z}_\ell$-modules on X (or ℓ-adic sheaf) is a family $\{\mathcal{M}_n, f_{n+1}: \mathcal{M}_{n+1} \to \mathcal{M}_n\}_n = \mathcal{M}$ such that

- $\mathcal{M}_n$ is a constructible sheaf of $\mathbb{Z}/\ell^n\mathbb{Z}$-modules
- $f_{n+1}: \mathcal{M}_{n+1} \to \mathcal{M}_n$ induces $\mathcal{M}_{n+1}/\ell^n \mathcal{M}_{n+1} \xrightarrow{\sim} \mathcal{M}_n$.

In the same way as before, we obtain a sequence of sheaves $0 \to \mathcal{M}_s \to \mathcal{M}_{n+s} \to \mathcal{M}_n \to 0$. From which

Def: $\mathcal{M}$ is flat if that sequence is exact $\forall n, s$.

Def: Étale cohomology of sheaves of $\mathbb{Z}_\ell$-modules:

$\mathcal{M}$ sheaf of $\mathbb{Z}_\ell$-modules

$$H^q(X, \mathcal{M}) := \varprojlim_n H^q(X, \mathcal{M}_n)$$ with $\mathcal{M}_n$ with it structure of sheaf of $\mathbb{Z}/\ell^n\mathbb{Z}$-mod's

$$H^q_c(X, \mathcal{M}) := \varprojlim_n H^q_c(X, \mathcal{M}_n)$$

[In particular: $H^q(X, \mathbb{Z}_\ell) = \varprojlim_n H^q(X, \mathbb{Z}/\ell^n\mathbb{Z})$

with $\mathbb{Z}_\ell$ be the sheaf of $\mathbb{Z}_\ell$-modules having $\mathcal{M}_n$ the constant sheaf $\mathbb{Z}/\ell^n\mathbb{Z}$]

Thm: $\mathcal{M} = (\mathcal{M}_n)_n$ flat sheaf of $\mathbb{Z}_\ell$-modules on X/k. Assume $k = k^{sep}$, and either X complete or $\ell \nmid \operatorname{char} k$.
Then $H^q(X, \mathcal{M})$ is fin. gen. and $\exists$ l.e.s.

$$\dots \to H(X, \mathcal{M}) \xrightarrow{\ell^n} H^q(X, \mathcal{M}) \to H^q(X, \mathcal{M}_n) \to H^q(X, \mathcal{M}) \to$$

Pf: For every $s \geq 0$ we have

$$0 \to \mathcal{M}_s \to \mathcal{M}_{n+s} \to \mathcal{M}_n \to 0$$

compatible with each others:

this is with $s = s+1$

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{M}_{s+1} & \to & \mathcal{M}_{n+1+s} & \to & \mathcal{M}_n & \to & 0 \\ & & \downarrow f_{s+1} & & \downarrow f_{n+1+s} & & \downarrow id & & \\ 0 & \to & \mathcal{M}_s & \to & \mathcal{M}_{n+1+s} & \to & \mathcal{M}_n & \to & 0 \end{array}$$

commutes

For each s, we hav l.e.s in cohomology

$$\begin{array}{l} H^n(\mathcal{M}_{s}) \to H^n(\mathcal{M}_{s+n}) \to H^n(\mathcal{M}_n) \to H^{n+1}(\mathcal{M}_s) \to \dots \\ \downarrow \\ H^n(\mathcal{M}_{s-1}) \to \quad \dots \\ \downarrow \end{array}$$

We can take limit over s on each column, and we obtain

l.e.s. $\dots \to H^n(\mathcal{M}) \to H^n(\mathcal{M}) \to H^n(\mathcal{M}_n) \to H^{n+1}(\mathcal{M}) \to \dots$

The limit is in general not an exact functor, but the ~~columns~~ columns satisfy the Mittag-Leffler conditions (as those cohomology grps are all modules of finite lenght). So, for all n, there is the s.e.s. $0 \to H^n(\mathcal{M})/\ell^m H^n(\mathcal{M}) \to H^n(\mathcal{M}_m) \to H^{n+1}(\mathcal{M})_{\ell^m} \to 0$, where $H^{n+1}(\mathcal{M})_{\ell^m}$ is the subgrp of elements divisible by ℓ^m (the maps are defined by the previous exact sequence, and the exactness is immediate to check).

Now take the limits over m (transition functions on the last column are multiplication by ℓ)

~~The first~~ • the last column vanish: $\lim H^{n+1}(\mathcal{M})_{\ell^m}$ gives the group of elemts in $H^{n+1}(\mathcal{M})$ that are divisible by all powers of ℓ, but $H^{n+1}(\mathcal{M})$ is by definition a limit of ℓ-power torsion finite groups, so no elements but 0 are divisible by all powers of ℓ.

• The second column is $H^n(\mathcal{M})$ by definition.

So, we conclude that $H^n(\mathcal{M}) \simeq \varprojlim_m H^n(\mathcal{M})/\ell^m H^n(\mathcal{M})$

But in this way we can recover the categorical equivalence for finite generated $\mathbb{Z}_\ell$-modules by setting $M_n := H^q(\mathcal{M}) / \ell^n H^q(\mathcal{M})$, and check the properties.

We conclude $H^q(\mathcal{M})$ finite generated $\mathbb{Z}_\ell$-module. $\square$

§ Sheaves of $\mathbb{Q}_\ell$-modules

Def: A $\mathbb{Q}_\ell$-sheaf is a $\mathbb{Z}_\ell$-sheaves $\mathcal{M} = (\mathcal{M}_n)$, but we define its cohomology as

$$H^q(X, \mathcal{M}) = \left(\varprojlim_n H^q(X, \mathcal{M}_n)\right) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$$

Exmp: $H^q(X, \mathbb{Q}_\ell) = \varprojlim_n H^q(X, \mathbb{Z}/\ell^n\mathbb{Z}) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell =$

$$= H^q(X, \mathbb{Z}_\ell) \otimes \mathbb{Q}_\ell$$