Talk 8: Poincare Duality and Lefschetz Fixed Point Formula

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These are notes for a talk given in the PhD-seminar on *the Weil Conjectures* in the Winter Term 2023/24 at the University of Duisburg-Essen. The main references for this talk are [Mil13] and [Mil80]. I would like to thank the organisers Giulio Marazza and Guillermo Gamarra for discussing numerous questions with me.

In this talk, we are continuing to prove properties of étale cohomology that look like properties of singular cohomology for complex varieties. The Goal of this talk is twofold: Firstly, we want to see a proof of Poincaré duality for étale cohomology. Secondly, we want to see the Lefschetz fixed point formula, which allows one to count the number of fixed points using cohomology.

Let k be an algebraically closed field throughout the talk.

1 Poincaré Duality

The classical statement of Poincare Duality constructs for an m-dimensional, oriented, connected manifold U a canonical isomorphism $H^r_c(U, \mathbb{Z}/n\mathbb{Z}) \to H_{m-r}(U, \mathbb{Z}/n\mathbb{Z})$. Using the duality between homology and cohomology, one can transform this into a perfect pairing

$$H^r_c(U, \mathbb{Z}/n\mathbb{Z}) \times H^{m-r}(U, \mathbb{Z}/n\mathbb{Z}) \to H^m_c(U, \mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/n\mathbb{Z}$$

The last isomorphism is induced by the choice of an orientation. When looking at the complex picture, one can note that (by a choice of $\sqrt{-1}$) the topological space $\mathbb{C}$ is already oriented and therefore all complex manifolds canonically carry an orientation. This yields a Poincaré duality statement for all connected complex manifolds. (Note that since a d-dimensional complex manifold has real dimension $m = 2d$, we will land in the cohomology group of degree $2d$.)

If we now want to move this idea to the world of étale cohomology, we therefore want to see two statements: We want to find a canonical isomorphism like

$$H^{2d}_c(X, \mathbb{Z}/n\mathbb{Z}) \to \mathbb{Z}/n\mathbb{Z}$$

and we want to have a perfect pairing like

$$H^r_c(X, \mathbb{Z}/n\mathbb{Z}) \times H^{2d-r}(X, \mathbb{Z}/n\mathbb{Z}) \to H^{2d}_c(X, \mathbb{Z}/n\mathbb{Z}).$$

In practice, things get a bit more general (we have more general sheaves) and the above statements change slightly, we introduce a twist, in order to obtain an isomorphism with $\mathbb{Z}/n\mathbb{Z}$ on the right.

Fix an integer $n \geq 1$ prime to the characteristic of k. let $\Lambda := \mathbb{Z}/n\mathbb{Z}$ and let $\Lambda(m) := \mu_n^{\otimes m}$. Last time, we constructed the cycle class map $\mathrm{cl}_X\colon \mathrm{CH}^*(X) \to H^{2*}(X, \Lambda(*))$ for X a non-singular variety. The elements of $\mathrm{CH}^i(X)$ are formal sums of codimension i irreducible closed subvarieties of X up to rational equivalence. The class $[Z]$ of a smooth, closed $i\colon Z \hookrightarrow X$ is send to the element $i_*(1_Z) \in H^{2i}(X, \Lambda(i))$, where i_* is the Gysin map that we defined in Talk 5.

With this, we can state Poincaré duality.

Theorem 1.1 (Poincaré Duality, [Mil80, Ch VI, Thm. 11.1]). *Let X be a smooth separated variety of dimension d over k.*

(a) There is a unique map $\eta(X)\colon H_c^{2d}(X, \Lambda(d)) \to \Lambda$ such that $\mathrm{cl}_X(P) \mapsto 1$ for any closed point P of X; moreover, $\eta(X)$ is an isomorphism if X is connected.

(b) For any constructible sheaf F of Λ-modules on X, the canonical pairings

$$H_c^r(X, F) \times \mathrm{Ext}_X^{2d-r}(F, \Lambda(d)) \to H_c^{2d}(X, \Lambda(d)) \xrightarrow[\sim]{\eta(X)} \Lambda$$

are non-degenerate.

In order to prove (the first part of) the theorem, we need to prove two lemmas.

Lemma 1.2 ([Mil80, Ch. VI, Lemma 11.3]). *For any connected, separated X of dimension d over k, we have $H_c^{2d}(X, \Lambda(d)) \cong \Lambda$.*

Proof. The proof proceeds by induction on d. The case $d = 1$ has been treated in Talk 5.

Let $X_0 \subset X$ be an open, dense subset of X. Then, we have the long exact sequence

$$\cdots \to H_c^i(X_0, \Lambda(d)) \to H_c^i(X, \Lambda(d)) \to H_c^i(X \setminus X_0, \Lambda(d)) \to H_c^{i+1}(X_0, \Lambda(d)) \to \dots$$

associated with the short exact sequence

$$0 \to j_! j^* \Lambda(d) \to \Lambda(d) \to i_* i^* \Lambda(d) \to 0$$

with $j\colon X_0 \to X$ and $i\colon X \setminus X_0 \to X$ the immersions.

From this and from what we have learned about cohomological dimension in Talk 5 [Mil13, Thm. 15.1], we get that this induces an isomorphism $H_c^{2d}(X_0, \Lambda(d)) \to H_c^{2d}(X, \Lambda(d))$. Thus, we may replace X in the statement by X_0 and vice versa. Now, we can find an a diagram of the form

$$\begin{tikzcd} X & X_0 \arrow[l, hook'] \arrow[r, hook] \arrow[dr] & \overline{X}_0 \arrow[d, "\pi"] \\ & & S \end{tikzcd}$$

where the hooked arrows are open (and dense) immersions and π is projective with fibres of dimension 1. In particular, S has dimension $d-1$. So, without loss of generality, we can assume X to be this $\overline{X}_0$.

The Kummer exact sequence yields the exact sequence of sheaves over S

$$\begin{tikzcd} \dots \arrow[r] & \underline{\mathrm{Pic}}_{X/S} \arrow[r, "n"] \arrow[d, "\deg"] & \underline{\mathrm{Pic}}_{X/S} \arrow[r] \arrow[d, "\deg"] & R^2\pi_*\Lambda(1) \arrow[r] & \dots \\ & \mathbb{Z} \arrow[r, "\cdot n"] & \mathbb{Z} \arrow[r] & \Lambda \arrow[r] & 0 \end{tikzcd}$$

The map $\underline{\mathrm{Pic}}_{X/S} \to \Lambda$ factors through $R^2\pi_*\Lambda(1)$. Indeed, it is obvious that the map factors through the image of $\underline{\mathrm{Pic}}_{X/S}$ in $R^2\pi_*\Lambda(1)$. By checking on fibres (remember, they are curves, so we have computed the situation explicitly in Talk 5), we see that it actually factors through all of $R^2\pi_*\Lambda(1)$. Thus, we get a canonical isomorphism $R^2\pi_*\Lambda(d) \cong \Lambda(d-1)$.

Now, a spectral sequence argument shows that

$$H_c^{2d}(X, \Lambda(d) \cong H^{2d-2}(S, R^2\pi_*\Lambda(d)) \cong H^{2d-2}(S, \Lambda(d-1)) \cong \Lambda,$$

where we get the last isomorphism using the induction hypothesis. ∎

Lemma 1.3 ([Mil80, Ch. VI, Lemma 11.4]). *Let $\pi\colon Y \to X$ be a separated, étale morphism where X is a smooth, separated variety of dimension d over k. Let P be a closed point of Y and let $Q = \pi(P)$. Then, the map*

$$\pi_*\colon H_c^{2d}(Y, \Lambda(d)) \to H_c^{2d}(X, \Lambda(d))$$

induced by the map $R_c^0\pi_*\Lambda(d) = \pi_!\pi^*\Lambda(d) \xrightarrow{\mathrm{tr}} \Lambda(d)$ sends $\mathrm{cl}_Y(P)$ to $\mathrm{cl}_X(Q)$.

Proof. Consider a compactification $Y \hookrightarrow \overline{Y} \xrightarrow{\bar{\pi}} X$ with $\bar{\pi}$ finite and inclusion $j\colon Y \to \overline{Y}$. Then, we can complete the defining morphisms for $\mathrm{cl}_Y(P)$ and $\mathrm{cl}_X(Q)$ to a commutative diagram

$$\begin{tikzcd}
H^{2d}_{\pi^{-1}(Q)}(\overline{Y}, j_!\Lambda(d)) \arrow[r] \arrow[d] & H^{2d}_P(\overline{Y}, j_!\Lambda(d)) \arrow[r] & H^{2d}_c(\overline{Y}, j_!\Lambda(d)) = H^{2d}_c(Y, \Lambda(d)) \arrow[d] \\
H^{2d}_Q(X, \Lambda(d)) \arrow[rr] & & H^{2d}_c(X, \Lambda(d))
\end{tikzcd}$$

in which both vertical maps are induced by $\mathrm{tr}\colon \bar{\pi}_* j_!\Lambda(d) \to \Lambda(d)$. Therefore, we only need to show that $s_{Q/X}$ and $s_{P/Y}$ have a common inverse image in $H^{2d}_{\pi^{-1}(Q)}(\overline{Y}, j_!\Lambda(d))$.

By excision, we can replace X with the spectrum of a strictly Henselian local ring. But then $\overline{Y}$ is only a discrete union of points and in this situation the statement trivially holds. ■

Proof of Thm. 1.1 (a). We shall prove this part of the theorem in two steps: First, we will construct η for $\mathbb{P}^d$, and then we shall reduce the general case to this case. We have seen last time [Mil80, Ch. VI, Ex. 9.7] that $H^{2d}(\mathbb{P}^d, \Lambda(d))$ is generated by $\mathrm{cl}_{\mathbb{P}^d}(P)$ for any closed point P of $\mathbb{P}^d$ and this class is independent of P. Thus, we have a unique choice for $\eta(\mathbb{P}^d)$.

Let X be as in the theorem. Then the smoothness of X yields a diagram of the form

$$\begin{tikzcd}
X & X_0 \arrow[l, hook', "j"'] \arrow[d, "\pi"] & \\
& U \arrow[r, hook, "j"] \arrow[d] & \mathbb{P}^d \arrow[dl] \\
& S = \operatorname{Spec} k &
\end{tikzcd}$$

in which both maps j are open immersions and π is étale and separated. (That π is separated follows from the fact that X, and therefore also X_0, are separated over S.) Thus, we can consider by the definition of compactly supported cohomology the isomorphisms

$$H_c^{2d}(X, \Lambda(d)) \xleftarrow{\sim} H_c^{2d}(X_0, \Lambda(d)) \xrightarrow{\pi_*} H_c^{2d}(U, \Lambda(d)) \xrightarrow{\sim} H^{2d}(\mathbb{P}^d, \Lambda(d)) \xrightarrow[\sim]{\eta(\mathbb{P}^d)} \Lambda.$$

By construction, for all closed points Q in U, the element $\mathrm{cl}_U(Q)$ is mapped to $1 \in \Lambda$. Accordingly, Lemma 1.3 yields that the same is true for any closed point P in X_0. Therefore, we can define $\eta(X)$ as the above composition. The map $\eta(X)$ clearly has the property for all closed $P \in X_0$. Furthermore, this map η is an isomorphism by Lemma 1.2, and it therefore is uniquely determined by the fact that it sends, for every closed point P of X_0, the class $\mathrm{cl}_X(P)$ to $1 \in \Lambda$. If we had used a different open X_0' to define $\eta(X)$, this open would have an intersection with X_0 as X is irreducible. Therefore, the map $\eta(X)$ sends all $\mathrm{cl}_X(P)$ to $1 \in \Lambda$ for P a closed point of X. Furthermore, all open subsets X_0 such that such a diagram exist construct the same map. ■

The proof of part (b) is based on the observation that if F is constructible then there exists $U \subseteq X$ open such that $F|_U$ is locally constant. Therefore if we can bootstrap the statement from an open subset (Step 3) and if we know it for locally constant sheaves (Step 7), we get the theorem. To prove the former, one reduces further to sheaves which are supported on a closed, smooth subscheme (Step 2). To prove the latter, one reduces to constant sheaves (Step 6) and then further to a situation similar to the one in the proof of Lemma 1.2 (Step 5). This is paired with an observation (Step 1) and a rather technical ingredient (Step 4). Furthermore, the entire proof is wrapped in an induction on the dimension of our space X.

Proof of Thm. 1.1 (b). We prove the statement by induction on the dimension d of X. We have already seen the case $d = 1$ in Talk 5. Let $\varphi^r(X, F)$ be the map $\mathrm{Ext}_X^{2d-r}(F, \Lambda(d)) \to H_c^r(X, F)^\vee$ induced by the pairing in the theorem.

Since we already now the beginning of the induction, we can assume that $\varphi^r(X, F)$ is an isomorphism whenever $\dim X \leq d - 1$; and we now need to prove that it is an isomorphism whenever $\dim X = d$.

Step 1. Let $\pi\colon X' \to X$ be a finite étale map, where X' and X are varieties as in Theorem 1.1. Then, for a sheaf F on X', we have that $\varphi^r(X', F)$ is an isomorphism if and only if $\varphi^r(X, \pi_* F)$ is an isomorphism.

Proof. The cohomology and ext group computations do not change under the pushforward of a finite étale map as π_* is exact and preserves injectives. Therefore, we at most need to worry about the square

$$\begin{tikzcd} H_c^{2d}(X', \Lambda(d)) \arrow[r, "\eta(X')"] \arrow[d, "\mathrm{tr}"] & \Lambda \arrow[d, equal] \\ H_c^{2d}(X, \Lambda(d)) \arrow[r, "\eta(X)"] & \Lambda \end{tikzcd}$$

commuting. But this follows from Lemma 1.3. #

Step 2. $\varphi^r(X, F)$ is an isomorphism if F has support on a smooth closed subvariety $Z \neq X$ of X.

Proof. We can regard F as a sheaf on Z. Because we already know that $\varphi^r(Z, F)$ is an isomorphism by the induction hypothesis, we have to show that there exists a commutative diagram

$$\begin{tikzcd} H_c^r(Z, F) & \times & \mathrm{Ext}_Z^{2a-r}(F, \Lambda(a)) \arrow[r] \arrow[d, "\cong"', "i_*"] & H_c^{2a}(Z, F) \arrow[r, "\eta(Z)"] \arrow[d, "\cong"', "i_*"] & \Lambda \arrow[d, equal] \\ H_c^r(X, i_* F) \arrow[u, "i^*", "\cong"'] & \times & \mathrm{Ext}_X^{2d-r}(i_* F, \Lambda(d)) \arrow[r] & H_c^{2d}(X, i_* F) \arrow[r, "\eta(X)"] & \Lambda, \end{tikzcd}$$

where a is the dimension of Z.

Consider a compactification $\overline{X}$ of X, and let $\overline{Z}$ be the closure of Z in $\overline{X}$, that is we have the square

$$\begin{tikzcd} Z \arrow[r, hook, "i"] \arrow[d, hook, "j"] & X \arrow[d, hook, "j"] \\ \overline{Z} \arrow[r, "i"] & \overline{X}. \end{tikzcd}$$

Then, we have $i_*j_! = j_!i_*$ and therefore constructing a diagram as above is equivalent to constructing a diagram

$$\begin{tikzcd} H^r(\overline{Z}, j_!F) & \times & \mathrm{Ext}^{2a-r}_Z(F, \Lambda(a)) \arrow[r] & H^{2a}(\overline{Z}, j_!F) \arrow[r, "\eta(Z)"] & \Lambda \arrow[d, equal] \\ H^r(\overline{X}, i_*j_!F) \arrow[u, "i^*", "\cong"'] & \times & \mathrm{Ext}^{2d-r}_X(i_*F, \Lambda(d)) \arrow[from=u, "\cong"', "i_*"] \arrow[r] & H^{2d}(\overline{X}, i_*j_!F) \arrow[from=u, "\cong"', "i_*"] \arrow[r, "\eta(X)"] & \Lambda. \end{tikzcd}$$

This can be done using similar tools as the ones the we used to show properties of the cycle class map. #

Step 3. Let U be an open subvariety of X. Then, $\varphi^r(X, F)$ is an isomorphism for all r if and only if $\varphi^r(U, F|_U)$ is an isomorphism for all r.

In particular, this step implies that it is enough to show that the theorem holds for finite locally constant sheaves.

Proof. One can construct a sequence $X = X_0 \supset X_1 \supset X_2 \supset \cdots \supset X_s = U$ of open immersions such that $Z_i := X_i \setminus X_{i+1}$ is smooth. If we denote the immersions by $j\colon X_{i+1} \to X_i$, and $i\colon Z_i \to X_i$, we have the exact sequence

$$0 \to j_!j^*F|_{X_i} \to F|_{X_i} \to i_*i^*F|_{X_i} \to 0.$$

Now the domain and codomain of φ^r have a long exact sequence associated with this sequence and φ^r is compatible with these long exact sequences. As we already know the Theorem for i_*i^*F by Step 2, we obtain that $\varphi^r(X_i, F|_{X_i})$ is an isomorphism for all F if and only if $\varphi^r(X_{i+1}, F)$ is an isomorphism for all r using a five-lemma argument. #

Step 4. Let X be a variety for which Poincaré Duality holds, and let $F^\bullet$ be a complex of sheaves on X that is bounded below and such that $\mathrm{H}^r(F^\bullet)$ is zero for $r >> 0$ and constructible for all r. Then, there are canonical non-degenerate pairings

$$\mathbb{H}^r_c(X, F^\bullet) \times \mathbb{E}\mathrm{xt}^{2d-r}_X(F, \Lambda(d)) \to H^{2d}_c(X, \Lambda(d)) \xrightarrow[\sim]{\eta} \Lambda.$$

Proof. We truncate $F^\bullet$ below and start truncating at r with $H^{r+1}(X, F) = 0$. Then, we use induction and an exact sequence relating the truncations. We then use the 5-lemma to deduce Step 4 from Poincaré Duality on X and an inductive argument. #

Step 5. Let S be a smooth, separated variety of dimension $d-1$ over k, and let $\pi\colon X \to S$ be projective and smooth with all fibres of dimension 1. Then $\varphi^r(X, F)$ is an isomorphism for any locally constant, constructible sheaf F on X.

Proof. The idea of this step is to relate the statement for F to the statement in Step 4. Since π is projective, we can construct a diagram

$$\begin{tikzcd} X \arrow[r, "\pi"] \arrow[d, hook, "j"] & S \arrow[r] \arrow[d, hook, "j"] & \mathrm{Spec}\, k, \\ \overline{X} \arrow[r, "\overline{\pi}"] & \overline{S} \arrow[ur] & \end{tikzcd}$$

in which $\overline{\pi}$ is projective, $\overline{S}$ and $\overline{X}$ are compactifications of S and X, respectively, and $\overline{\pi}^{-1}(S) = X$.

Now, let $j_!\Lambda(d) \to \Lambda_!^*$ and $j_!\Lambda_S(d-1) \to \Lambda_{S!}^*$ be injective resolutions. We can now consider

$$\begin{tikzcd}
\mathrm{Hom}_{\overline{X}}(\Lambda, F_!^*[r]) \arrow[d, "\cong"] & \times & \mathrm{Hom}_{\overline{X}}(F_!^*[r], \Lambda_![2d]) \arrow[r] \arrow[d] & \mathrm{Hom}_{\overline{X}}(\Lambda, \Lambda_![2d]) \arrow[d, "\cong"] \\
\mathrm{Hom}_{\overline{S}}(\Lambda, \overline{\pi}_* F_!^*[r]) & \times & \mathrm{Hom}_{\overline{S}}(\overline{\pi}_* F_!^*[r], \overline{\pi}_*\Lambda_!^*[2d]) \arrow[r] \arrow[d] & \mathrm{Hom}_{\overline{S}}(\Lambda, \overline{\pi}_*\Lambda_!^*[2d]) \arrow[d] \\
 & & \mathrm{Hom}_{\overline{S}}(\overline{\pi}_* F_!^*[r], \Lambda_{S!}^*[2d-2]) \arrow[r] & \mathrm{Hom}_{\overline{S}}(\Lambda, \Lambda_{S!}^*[2d-2]),
\end{tikzcd}$$

where $j_!F \to F_!^*$ is an injective resolution of $j_!F$. If we now pass to homotopy classes, we get the comparison diagram

$$\begin{tikzcd}
H_c^r(X, F) \arrow[d, "\cong"] & \times & \mathrm{Ext}_X^{2d-r}(F, \Lambda(d)) \arrow[r] \arrow[d, "\beta"] & H_c^{2d}(X, \Lambda(d)) \arrow[r, "\eta(X)", "\cong"'] \arrow[d, "\gamma"] & \Lambda \\
\mathbb{H}_c^r(S, R\pi_*F) & \times & \mathbb{E}\mathrm{xt}_S^{2d-r}(R\pi_*F, \Lambda(d-1)) \arrow[r] & H_c^{2d}(S, \Lambda(d-1)) \arrow[r, "\eta(S)", "\cong"'] & \Lambda,
\end{tikzcd}$$

where $R\pi_*F = \pi_*F^*$ and $F \to F^*$ is an injective resolution of F.

We know by Step 4 that the bottom pairing is perfect. By the characterisation of the map η and Lemma 1.3, we know that γ is an isomorphism. Therefore, in order to prove Step 5, we only have to prove that β is an isomorphism.

We do this as follows: First, we sheafify β to obtain a map $\tilde{\beta}\colon \pi_*\underline{\mathrm{Hom}}_X(F, \Lambda(1)^*) \to \underline{\mathrm{Hom}}_S(R\pi_*F, \Lambda^*[-2])$. This is done using the usual base change trick to construct a morphism of sheaves. When taking cohomology of these $\tilde{\beta}$ we retrieve β, up to a Tate twist.

Both of these sheaves are flasque and therefore it is enough to show that $\tilde{\beta}$ is a quasi-isomorphism, which can be done on stalks. If we look at the fibre of a point $s \in S$ on X, we get a curve. Therefore, after some identifications, proving that $\tilde{\beta}_s$ is a quasi-isomorphism boils down to Poincaré Duality for curve, which we have seen in Talk 5. #

Step 6. If F is constant, then $\varphi^r(X, F)$ is an isomorphism.

Proof. One can show [Mil80, Ch. III, Lemma 3.13] that we can manoeuvrer ourselves in the following situation: We can find an open dense subvariety $X_0 \subset X$ and maps

$$\begin{tikzcd}
X & X_0 \arrow[l, hook'] \arrow[r, hook] \arrow[dr] & \overline{X}_0 \arrow[d, "\pi"] \\
 & & S
\end{tikzcd}$$

such that $\pi\colon \overline{X}_0 \to S$ satisfies the hypothesis of Step 5. Now, there exists a constant sheaf $\overline{F}$ on $\overline{X}_0$ such that $F|_{X_0} = \overline{F}|_{X_0}$. Therefore by Step 3, it suffices to prove that $\varphi(\overline{X}_0, \overline{F})$ is an isomorphism for all r, but this follows form Step 5. #

Step 7. If F is locally constant, then $\varphi^r(X, F)$ is an isomorphism.

Proof. We can do the same trick that we also did in Talk 5: There is a finite étale map $\pi\colon X' \to X$ such that $F|_{X'}$ is constant. Now, we can, as in Talk 5, look at the short exact sequence

$$0 \to F \to \pi_*(F|X') \to F_1 \to 0.$$

We know by Steps 1 and 6 that the Theorem holds for $\pi_*(F|X')$. One can now proceed by induction on r to show that $\varphi^r(X,F)$ is an isomorphism. For this one needs to note that F_1 is also locally constant, at least if we chose π wisely. #

An application of Step 3 to the locally constant sheaf F now completes the proof. ■

2 Intermezzo: The Gysin Map

We can use Poincaré duality to construct proper pushforwards in étale cohomology. This is a generalisation of the maps in the Gysin sequence, which we saw in Talk 5.

Let $\pi\colon Y \to X$ be a proper map of smooth separated varieties over k, let $a = \dim X$ and $d = \dim Y$, and $e = d - a$. Then, we have a restriction map

$$\pi^*\colon H_c^{2d-r}(X, \Lambda(d)) \to H_c^{2d-r}(Y, \Lambda(d)),$$

which, by Poincaré duality, yields a map

$$\pi_*\colon H^r(Y, \Lambda) \to H^{r-2e}(X, \Lambda(-e)).$$

We call π_* the *Gysin map*.

Remark 2.1 ([Mil13, Rmk. 24.2]). (a) The map π_* is uniquely determined by the equation

$$\eta_X(\pi_*(y) \cup x) = \eta_Y(y \cup \pi^*(x))$$

for $x \in H_c^{2d-r}(X, \Lambda(d))$ and $y \in H^r(Y, \Lambda)$. Indeed, this is just a spelled out version of the definition.

(b) If π is a closed immersion $Y \hookrightarrow X$, then π_* is the Gysin map defined in Talk 5. In this case, $-e$ is the codimension of Y in X. In particular, we have $\pi_*(1_Y) = \mathrm{cl}_X(Y)$, where 1_Y is the identity element in $H^0(Y, \Lambda) = \Lambda$. (One can see this by examining Step 2 in the proof of Poincaré duality.)

(c) The proper pushforward is functorial, i.e. we have

$$(\pi_1)_* \circ (\pi_2)_* = (\pi_1 \circ \pi_2)_*.$$

Indeed, we can use the definition and fill in that $\pi_2^* \circ \pi_1^* = (\pi_1 \circ \pi_2)^*$ and use the naturality of η.

(d) If Y and X are complete, we have

$$\eta_X(\pi_*(y)) = \eta_Y(y)$$

for $y \in H^{2d}(Y, \Lambda(d))$, because

$$\eta_X(\pi_*(y)) = \eta_X(\pi_*(y) \cup 1_X) = \eta_Y(y \cup \pi^*(1_X)) = \eta_Y(y \cup 1_Y) = \eta_Y(y).$$

(e) (Projection formula) If Y and X are complete, then we have

$$\pi_*(y) \cup x = \pi_*(y \cup \pi^*(x))$$

for $x \in H^r(X)$ and $y \in H^s(Y)$. One can think of this as saying π_* is a $H^*(X)$-linear map if we equip $H^*(Y)$ with the $H^*(X)$-module structure induced by $\pi^*\colon H^*(X) \to H^*(Y)$. One can prove this by using the determining equation, that is, one computes for $x' \in H_c^{2d-r}(X, \Lambda(d)) = H^{2d-r}(X, \Lambda(d))$ that

$$\eta_X(\pi_*(y \cup \pi^*(x)) \cup x') = \eta_Y(y \cup \pi^*(x) \cup \pi^*(x')) = \eta_Y(y \cup \pi^*(x \cup x')) = \eta_X(\pi_*(y) \cup x \cup x').$$

From the perfectness of the pairing, we get the desired identity.

(f) If $\pi\colon Y \to X$ is a finite map of degree δ, we have $\pi_* \circ \pi^* = \delta$. This can be seen by examining the behaviour of π_* and π^* on the top and bottom cohomology, where we know explicitly what is happening.

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3 Lefschetz Fixed Point Formula

Let l be a prime different from the characteristic of k.

The Lefschetz Fixed-Point Theorem counts the number of fixed points (with multiplicities) of a given map. This is done in terms of a formula that is similar to an Euler characteristic but with traces of the map instead of the dimensions of the cohomology groups. In order to understand this, we have to briefly review the definition of the trace:

Let V be a finite-dimensional vector space, and let $\varphi\colon V \to V$ be a linear map. If (a_{ij}) is the matrix of φ with respect to a basis (e_i) of V, then the trace of φ, denoted by $\operatorname{Tr}(\varphi|V)$, is defined as $\sum_i a_{ii}$. The trace is independent of the choice of basis. We can also characterise this using the dual basis. (This will come in handy later.) Let (f_i) be the dual basis of (e_i) in the dual vector space $V^\vee$. If $\cdot$ denotes the evaluation, we have that $e_i \cdot f_j = \delta_{ij}$, where δ_{ij} is the Kronecker delta. Therefore we have that

$$\sum_i \varphi(e_i) \cdot f_i = \sum_i (\sum_j a_{ji} e_j) \cdot f_i = \sum_i a_{ii} = \operatorname{Tr}(\varphi|V).$$

In order to make sense of the statement of the Lefschetz fixed point formula, let us first take a look at what counting fixed points actually means: If we have a morphism $\varphi\colon X \to X$, we are looking for points $x \in X$ with $\varphi(x) = x$. These points are characterised by lying in the intersection $\Gamma_\varphi \cap \Delta \subset X \times X$ of the graph Γ_φ of φ and the diagonal Δ. Similar to counting zeroes of a polynomial, naive point counting is not really a nice invariant. Instead, we want to count points with multiplicities. Last time, we introduced the intersection product to do exactly that. Thus, if X has dimension n, we are considering $\Gamma_\varphi \cdot \Delta \in \mathrm{CH}^{2n}(X \times X) \cong \mathbb{Z}$.

In order to compute this number, we want to (in the spirit of the seminar) relate it with étale cohomology. In the last talk, we have also defined the cycle class map $\mathrm{cl}_X\colon \mathrm{CH}^i(X) \to H^{2i}(X, \mathbb{Q}_l)$. If X is complete then Poincaré duality yields the commutative diagram

$$\begin{tikzcd}
\mathrm{CH}^{2n}(X \times X) \arrow[r, "\mathrm{cl}_{X\times X}"] \arrow[dr, "\cong"'] & H^{4n}(X \times X) \arrow[r, "\eta_{X\times X}"] & \mathbb{Q}_l \\
& \mathbb{Z} \arrow[ur] &
\end{tikzcd}$$

Thus, we can try to compute $\mathrm{cl}(\Gamma_\varphi \cdot \Delta)$ and then go further to the image of the integers in $\mathbb{Q}_l$. After some computations, which we shall see shortly, one arrives at the following result.

Theorem 3.1 (Lefschetz Fixed-Point Formula, [Mil13, Theorem 25.1]). *Let X be a compete non-singular variety over k, and let $\varphi\colon X \to X$ be a regular map. Then*

$$(\Gamma_\varphi \cdot \Delta) = \sum (-1)^r \operatorname{Tr}(\varphi | H^r(X, \mathbb{Q}_\ell)),$$

where Γ_φ is the graph of φ and Δ is the diagonal in $X \times X$, and $(-)$ means applying the isomorphism $\mathrm{CH}^{2\dim X}(X \times X) \cong \mathbb{Z}$.

Remark. In topology, one can rather easily prove the existence of a fixed point using that a fixed-point free map is homotopic to a simplicial fixed-point free map after refining the simplicial structure. Then the proof leverages, just as diffrent compuational methods for computing the

Euler Charactersitic that the trace of the homology groups of a complex and the trace of the map on a complex level are related. With the, one move the computation to cohomology which is a nice and subtle move. ⌟

Remark. In the process of stating the Lefschetz Fixed-Pint Formula, we have used some results and constructions that we know for cohomology with $\mathbb{Z}/n\mathbb{Z}$ also for cohomology with $\mathbb{Q}_l$ coefficients. This is okay, as one can extend the results that we have seen to the case of $\mathbb{Q}_l$ by using naturality statements. We illustrate how this is done by proving the statement that for a $2d$-dimensional, complete scheme X over k, we have a canonical homomorphism (which is in fact an isomorphism) $\eta(X)\colon H^{2d}(X, \mathbb{Q}_l) \to \mathbb{Q}_l$: By Poincaré duality 1.1, we have for each n a unique homomorphism $H^{2d}(X, \mathbb{Z}/l^n\mathbb{Z}) \to \mathbb{Z}/l^n\mathbb{Z}$ defined by sending $\mathrm{cl}_X(P)$ to 1 for any point $P \in X$. This construction is compatible with the canonical maps

$$\begin{tikzcd} H^{2d}(X, \mathbb{Z}/l^n\mathbb{Z}) \arrow[r, "\eta(X)"] & \mathbb{Z}/l^n\mathbb{Z} \\ H^{2d}(X, \mathbb{Z}/l^{n+1}\mathbb{Z}) \arrow[r, "\eta(X)"] \arrow[u] & \mathbb{Z}/l^{n+1}\mathbb{Z}, \arrow[u] \end{tikzcd}$$

which we can see using the description using $\mathrm{cl}_X(P)$. By taking the inverse limit on both sides and tensoring with $\mathbb{Q}_l$, we get the desired homomorphism $\eta(X)\colon H^{2d}(X, \mathbb{Q}_l) \to \mathbb{Q}_l$. ⌟

In order to get the theorem, we first need to find a description of the cycle class of the graph of φ in $H^*(X \times X, \mathbb{Q}_l)$. By the Künneth Theorem, we have $H^*(X \times X, \mathbb{Q}_l) \cong H^*(X, \mathbb{Q}_l) \otimes_{\mathbb{Q}_l} H^*(X, \mathbb{Q}_l)$. We can use this description to calculate the image.

Lemma 3.2 ([Mil13, Lemma 25.4]). *Let (e_i) be a basis of $H^*(X)$, and let (f_i) be the basis of $H^*(X)$ that is dual relative to the cup-product, so that $e_i \cup f_i = \delta_{ij} e^{2d}$ (here δ_{ij} is the Kronecker delta and e^{2d} the canonical generator of H^{2d}). For any morphism of schemes $\varphi\colon X \to X$, we have*

$$\mathrm{cl}_{X \times X}(\Gamma_\varphi) = \sum_i \varphi^*(e_i) \otimes f_i.$$

Proof. The (f_i) form a basis of $H^*(X, \mathbb{Q}_l)$ as a $\mathbb{Q}_l$-vector space. Therefore, they form a basis of $H^*(X \times X) \cong H^*(X, \mathbb{Q}_l) \otimes_{\mathbb{Q}_l} H^*(X, \mathbb{Q}_l)$ as a $H^*(X, \mathbb{Q}_l)$-module. Thus, we can write

$$\mathrm{cl}_{X \times X}(\Gamma_\varphi) = \sum_i a_i \otimes f_i$$

for unique elements $a_i \in H^*(X, \mathbb{Q}_l)$. By the next lemma, we have

$$\begin{aligned} \varphi^*(e_j) &= p_*((\sum_i a_i \otimes f_i) \cup (1 \otimes e_j)) \\ &= p_*(\sum_i a_i \otimes (f_i \cup e_j)) \\ &= p_*(a_j \otimes e^{2d}) = a_j, \end{aligned}$$

where $p\colon X \times X \to X$ is the projection on the first component. (The last assertion can be seen using the definition of p_* and the projection formula.) ■

It remains to prove the relation between φ^* and $\mathrm{cl}_{X \times X}(\Gamma_\varphi)$.

Lemma 3.3 ([Mil13, Lemma 25.3]). *For any morphisms of schemes $\varphi\colon X \to Y$ between complete schemes X and Y over k and any $y \in H^*(Y)$, we have*

$$p_*(\mathrm{cl}_{X \times Y}(\Gamma_\varphi) \cup q^* y) = \varphi^*(y),$$

where $p, q\colon X \times X \to X$ denote the projections.

Proof. This is essentially a formal consequence of the properties of the pushforward, that is the statement follows from Remark 2.1:

$$\begin{aligned}
p_*(\mathrm{cl}_{X\times Y}(\Gamma_\varphi) \cup q^*y) &= p_*((1,\varphi)_*(1_{X\times X}) \cup q^*y) \\
&= p_*(1,\varphi)_*(1_{X\times X} \cup (1,\varphi^*)q^*y) \\
&= (p\circ(1,\varphi))_*(1 \cup (q\circ(1,\varphi^*))^*y) \\
&= \mathrm{id}_*(1\cup\varphi^*y) = \varphi^*(y). \qquad \blacksquare
\end{aligned}$$

With this, we can prove the Lefschetz fixed point formula.

Proof of Theorem 3.1. Let e_i^r be a basis for $H^r(X)$, and let f_i^{2d-r} be the dual basis for $H^{2d-r}(X)$. Then, we have by Lemma 3.2

$$\mathrm{cl}_{X\times X}(\Gamma_\varphi) = \sum_{r,i} \varphi^*(e_i^r) \otimes f_i^{2d-r}$$

and

$$\mathrm{cl}_{X\times X}(\Delta) = \mathrm{cl}_{X\times X}(\Gamma_{\mathrm{id}_X}) = \sum_{r,i} e_i^r \otimes f_i^{2d-r} = \sum_{r,i}(-1)^{r(2d-r)} f_i^{2d-r} \otimes e_i^r = \sum_{r,i}(-1)^r f_i^{2d-r} \otimes e_i^r.$$

Now, the cycle class map is compatible with products, and therefore we have

$$\begin{aligned}
\mathrm{cl}_{X\times X}(\Gamma_\varphi \cdot \Delta) &= \left(\sum_{r,i} \varphi^*(e_i^r) \otimes f_i^{2d-r}\right) \cdot \left(\sum_{r',j} (-1)^{r'} f_j^{2d-r'} \otimes e_j^{r'}\right) \\
&= \sum_{r,r',i,j} (-1)^{r'} (\varphi^*(e_i^r) f_j^{2d-r'}) \otimes (f_i^{2d-r} e_j^{r'}) \\
&= \sum_{r,i} (-1)^r \varphi^*(e_i^r) f_i^{2d-r} \otimes e^{2d} \\
&= \sum_r (-1)^r \operatorname{Tr}(\varphi^* | H^r(X,\mathbb{Q}_l))(e^{2d}\otimes e^{2d}).
\end{aligned}$$

If we now apply $\eta_{X\times X}$ to both sides, we get the result. $\blacksquare$

References

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