ESAGA Research Seminar

— Moduli Stacks of Formal Groups —

University of Duisburg-Essen, Winter Term 2025/26

Introduction: The recent papers [2] and [3] by Barthel, Schlank, Stapleton and Weinstein make significant progress on problems from algebraic topology related to the computation of stable homotopy groups of spheres. The seminar will provide an introduction to these articles with an emphasis on the methods from arithmetic geometry used by the authors.

To put this into perspective, fix a prime number p, an integer $n \geq 1$ and a one-dimensional commutative formal group Γ_n over $\overline{\mathbb{F}}_p$ which is of height n. The Morava stabilizer group of Γ_n is the profinite group $\mathbb{G}_n = \mathrm{Aut}_{\overline{\mathbb{F}}_p}(\Gamma_n) \rtimes \mathrm{Gal}(\overline{\mathbb{F}}_p)$ where the automorphism group $\mathrm{Aut}_{\overline{\mathbb{F}}_p}(\Gamma_n) \cong \mathfrak{o}_D^\times$ of Γ_n is isomorphic to the group of units of the maximal order of the central $\mathbb{Q}_p$-division algebra D of invariant $1/n$. One may also interpret $\mathbb{G}_n \cong \widehat{D^\times}$ as the profinite completion of $D^\times$.

By a famous result of Lubin and Tate, deformations of Γ_n to formal groups over $W = W(\overline{\mathbb{F}}_p)$ are parametrized by a formal scheme $\mathrm{LT}_n = \mathrm{Spf}(A)$ where $A \cong W[\![u_1, \ldots, u_{n-1}]\!]$. Let ω be the module of invariant differential forms of the universal deformation of Γ_n. This is a free A-module of rank 1 with a continuous action of $\mathbb{G}_n$. The relation to algebraic topology comes from a spectral sequence

$$(*) \qquad \mathrm{H}^s_{cts}(\mathbb{G}_n, \omega^{\otimes 2t}) \Longrightarrow \pi_{t-s} L_{K(n)} S^0$$

constructed by Devinatz and Hopkins. The right hand side denotes the homotopy groups of the $K(n)$-local sphere at p.

The articles of Barthel, Schlank, Stapleton and Weinstein focus on the left hand side of $(*)$ which is related to the cohomology of the quotient stack $[\mathrm{LT}_n/\mathbb{G}_n]$. The latter is isomorphic to the formal completion of the moduli stack of formal groups of height n. The first part of the seminar (Talks 2–8) will provide an introduction to these moduli stacks. This is well documented in two sets of lecture notes by Goerss and Lurie (cf. [8] and [11]).

The second part of the seminar (Talks 9–12) will provide some background on the motivating example from algebraic topology mentioned above. This includes the classical problem of computing stable homotopy groups of spheres and the entrance of formal groups through complex oriented multiplicative cohomology theories. We will also sketch the construction of the spectral sequence $(*)$.

The third part of the seminar (Talks 13–14) will give a very brief account of some of the more advanced techniques from arithmetic geometry used by Barthel, Schlank, Stapleton and Weinstein. This concerns the isomorphism between the Lubin-Tate and Drinfeld towers, as well as the recent results of Colmez, Dospinescu and Nizioł on the pro-étale cohomology of Drinfeld's symmetric space (cf. [4] and [5]).

For an easier accessibility, the program contains several links to overview articles on Wikipedia and nLab.

1. Introduction and overview [Oct. 16]: Give an overview of the seminar and of the individual talks.

2. Formal group laws and the Lazard ring [Oct. 23]: Recall the notion of a one-dimensional commutative formal group law. Introduce Lazard's graded ring L and its universal property. Show that $L \cong \mathbb{Z}[t_1, t_2, \ldots]$ is isomorphic to a polynomial ring in countably many variables over the integers. See [6], Sections 1.1–1.5; [10], Théorème II; [11], Lectures 2–3.

3. Formal Lie varieties and formal groups [Oct. 30]: Introduce formal Lie varieties and formal groups. Focus on the Zariski local structure as in [8], Remark 1.31 & Example 2.3 and [11], Definition 11.5. Introduce the prestack $\mathcal{M}_{\mathrm{FG}}$ of formal groups and show that this is a stack for the *fpqc*-topology (cf. [8], Proposition 1.36 & Proposition 2.6). See [1], Chapter 2; [8], §§1.3-2.1; [11], Lecture 11.

4. The moduli stack as a quotient [Nov. 6]: Introduce the affine group scheme $G = \mathrm{Spec}(\mathbb{Z}[b_0^{\pm 1}, b_1, b_2, \ldots])$ acting on $\mathrm{Spec}(L)$ by change of coordinates of formal group laws. Show that the morphism $\mathrm{Spec}(L) \to \mathcal{M}_{\mathrm{FG}}$ induced by the universal formal group law is a presentation of stacks inducing an isomorphism $[\mathrm{Spec}(L)/G] \cong \mathcal{M}_{\mathrm{FG}}$ (cf. [8], Theorem 2.30). If time permits explain why the more naive prestack of formal group laws fails to be a stack (cf. [11], Proposition 11.7 (1)). See [1], §2.4.3; [8], §§2.3–2.4; [11], Lecture 11.

5. The height stratification [Nov. 13]: Define the height of a formal group with respect to a prime number p and introduce the elements v_n, focusing on the explicit local case in [8], Remark 5.4 and [11], Definition 12.13. Introduce the closed substack $\mathcal{M}_{\mathrm{FG}}^{\geq n} \subset \mathcal{M}_{\mathrm{FG}} \times \mathrm{Spec}(\mathbb{Z}_{(p)})$ of formal groups of height at least n. Describe $\mathcal{M}_{\mathrm{FG}}^{\geq n}$ and the height-n stack $\mathcal{M}_{\mathrm{FG}}^{n} = \mathcal{M}_{\mathrm{FG}}^{\geq n} \backslash \mathcal{M}_{\mathrm{FG}}^{\geq n+1}$ in terms of $v_0, \ldots, v_n$ (cf. [11], Lecture 13). Explain why $\mathcal{M}_{\mathrm{FG}}^{0} \cong \mathcal{M}_{\mathrm{FG}} \times \mathrm{Spec}(\mathbb{Q}) \cong B\mathbb{G}_{m,\mathbb{Q}}$ (cf. [6], Lecture 1, Proposition 2, [8], Corollary 3.22 and [11], Corollary 12.3) and $\mathcal{M}_{\mathrm{FG}}^{\geq 1} \cong \mathcal{M}_{\mathrm{FG}} \times \mathrm{Spec}(\mathbb{F}_p)$. See [1], §2.4.3; [3], §3.1; [6], Lecture 1, §1.6; [8], §5.1; [11], Lectures 12–13.

6. The Morava stabilizer group [Nov. 20]: Explain the existence of a formal group Γ_n of height n over $\overline{\mathbb{F}}_p$ (cf. [11], Corollary 13.2). Sketch the proof of Lazard's uniqueness theorem in the form of [8], Theorem 5.23 or [11], Theorem 14.1. Introduce the Morava stabilizer group $\mathbb{G}_n$ and show that its classifying stack is isomorphic to $\mathcal{M}_{\mathrm{FG}}^{n}$ (cf. [8], Theorem 5.36 and [11], Proposition 19.1). Make the structure of $\mathbb{G}_n$ explicit. See [6], Lecture 1, §1.8–1.9; [7], Chapter III.2; [11], Theorems 13.10/11, Lectures 14 & 19.

7. The deformation space of Lubin-Tate [Nov. 27]: Fix Γ_n as above and introduce the deformation functor LT_n. Show that it is pro-represented by $A \cong W[\![u_1, \ldots, u_{n-1}]\!]$ where $W = W(\overline{\mathbb{F}}_p)$. Explain the action of $\mathbb{G}_n$ on LT_n and sketch the construction of the Lubin-Tate tower via Drinfeld level structures. See [6], Lecture 1, §§2–3, and Lecture 5; [8], §7.1; [11], Lecture 21

8. The formal completion of the height-n stack [Dec. 4]: Introduce the formal completion $\hat{\mathcal{M}}_{\mathrm{FG}}^{n}$ of the stack $\mathcal{M}_{\mathrm{FG}}^{n}$ (cf. [3], page 17 or [8], page 108). Give an account of the fact that the morphism $\mathrm{LT}_n \to \hat{\mathcal{M}}_{\mathrm{FG}}^{n}$ induced by the universal deformation is a pro-étale $\mathbb{G}_n$-torsor. In other words, we have $\hat{\mathcal{M}}_{\mathrm{FG}}^{n} \cong [\mathrm{LT}_n/\mathbb{G}_n]$ (cf. [8], Theorem 7.22 or [3], Proposition 3.1.2). See

[3], §§3.1–3.3; [8], §§7.2–7.3.

9. Stable homotopy groups of spheres [Dec. 11]: Recall the foundations of homotopy theory for topological spaces and sketch the proof of the Freudenthal suspension theorem. List a few classical results concerning (stable) homotopy groups of spheres and give some of the necessary techniques and arguments. See [9], §§4.1–4.2; [12], Chapters 9 & 11; [13], Chapter 1, §§1–2; [16], Chapter 6.

10. Generalized cohomology theories and spectra [Dec. 18]: Recall the definition of relative singular (co)homology and its formal properties. Define generalized cohomology theories via the Eilenberg-Steenrod axioms and explain the relation to spectra via the Brown representability theorem. Give examples including the Eilenberg-Maclane spectrum and the sphere spectrum. See [12], Chapters 18, 19 & 22; [15], Chapters 7–10; [16], Chapter 17.

11. Complex oriented cohomology theories [Jan. 8]: Define complex oriented multiplicative cohomology theories. Explain how complex orientations give rise to formal group laws (cf. [11], Proposition 4.10 or [13], Lemma 4.1.4). Define the complex bordism MU and state Quillen's fundamental theorem $\pi_*\mathrm{MU} \cong L$ (cf. [11], Theorem 7.1 or [13], Theorem 4.1.6). State Landweber's exact functor theorem: If M is a graded L-module such that $(p = v_0, \ldots, v_n)$ is an M-regular sequence for every p and n then $X \mapsto \mathrm{MU}^*(X) \otimes_L M$ is a generalized homology theory (cf. [11], Corollary 15.6 and Theorem 16.1). See [11], Lectures 1, 4-7 & 15; [13], §4.1.

12. The Devinatz-Hopkins spectral sequence [Jan. 15]: Define Morava E-theory and Morava K-theory. Sketch the construction of the Devinatz-Hopkins spectral sequence $(*)$ and state the chromatic splitting and vanishing conjectures in the rational form of [2], Theorems A and B. Explain how Theorem B implies Theorem A using Lazard's computation of the continuous cohomology of p-adic Lie groups. See [2], §§1–2 and §3.8; [11], Lecture 22.

13. The isomorphism between the Lubin-Tate and the Drinfeld tower [Jan. 22]: Give an account of the isomorphism between the Lubin-Tate and the Drinfeld tower (cf. [2], Theorem 3.9.1, [3], Theorem 3.7.3 or [14], Theorem 7.2.3). Sketch the proof of [2], Theorem B, as in [2], §3.9. Explain how it reduces to a statement controlling the pro-étale cohomology of the open unit ball and of Drinfeld's symmetric space (cf. [2], Theorem 3.9.4). See [2], §3.9; [3], §§3.1–3.7; [14], §7.

14. The pro-étale cohomology of Drinfeld's symmetric space [Jan. 29]: Explain how the continuous $\mathbb{G}_n$-cohomology of the (principal) units of A is related to the pro-étale cohomology of Drinfeld's symmetric space. Sketch the necessary computation of the pro-étale cohomology of Drinfeld's symmetric space and derive [3], Theorems C and D. See [3], §§1, 4 & 5.

15. [Feb. 5]: Programm discussion for the summer term 2026.

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