Hodge theory and mixed Hodge structures

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The purpose of this seminar is to to study some topics in Hodge theory. The main focus of the seminar will be Deligne's construction of mixed Hodge structures (MHS) on complex varieties. In following through this construction, we will see in action several different geometric or algebraic tools, like the general notions of Hodge complexes and their realization in the derived category of a smooth variety, compactifications, simplicial methods involving cohomological descent and hypercoverings.

The following is a tentative plan for the seminar.

- Briefly recall the construction of pure Hodge structures on smooth projective varieties, using the Khäler form and Dolbeault cohomology. State the most relevant consequences of the Hodge structure, like hard Lefschetz theorem and Lefschetz decomposition.
- Abstract introduction to Hodge theory. Study the properties of the categories of Hodge structures and of MHS's: being abelian, strictness, canonical decomposition, polarizations.
- Explain the construction of MHS's on smooth varieties. This involves first the introduction of the mixed Hodge complex as an object in $D(X)$, which recovers the Hodge structure on $H^*(X)$, where X is a smooth projective variety. Later, we consider a Normal Crossing Divisor Y in X and find a MHS on the complex of differential forms on X with logarithmic singularities along Y. The hypercohomology of this complex is isomorphic to the cohomology of $X - Y$, so we get MHS's on the cohomology of smooth varieties after taking a compactification.
- Finally, pass to the construction of MHS's on general complex varieties. This involves using hypercoverings and cohomological descent. Indeed, we first see how to find MHS's on smooth simplicial varieties and then take a logarithmic simplicial resolution $U_\bullet \to X$ to descend the MHS to a general variety X. Depending on the interest of the speakers, this last part could be done at different levels of depth. We could spend some time going into the machinery of cohomological descent, or we could just take it as a black box and only use the results needed for the construction of MHS's.

Of course, Hodge theory is a very wide and classical topic and there are several different parts of the theory that different people might find appealing. For example, other related topics that might be of interest are variations of (mixed) Hodge structures, which is the study of how Hodge structures change in families, or some results concerning the Hodge conjecture, like the Hodge conjecture for abelian varieties using the classification of Mumford-Tate groups. Depending on the time restrictions and on our interests, we could also choose to include these or other related topics, possibly by giving a quicker overview of the construction of MHS's.

The literature is also very wide, giving everyone the chance to choose the source and the approach they prefer. Just to have an idea, a somewhat concise explanation of the construction of MHS's as illustrated above can be found in [ET13]. A comprehensive account regarding the theory of MHS's and related topics can be found in [PS08], while [Cat+14] contains material for a more general study of several topics in Hodge theory.

References

[Cat+14] E. CATTANI et al. *Hodge Theory*. Mathematical Notes. Princeton University Press, 2014. ISBN: 9781400851478.

[ET13] Fouad ELZEIN and Lê Dung TRANG. *Mixed Hodge Structures*. 2013. arXiv: 1302.5811 [math.AG]. URL: https://arxiv.org/abs/1302.5811.

[PS08] Chris A. M. PETERS and Joseph H. M. STEENBRINK. *Mixed Hodge structures*. Vol. 52. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]. Springer-Verlag, Berlin, 2008, pp. xiv+470. ISBN: 978-3-540-77015-2.