

Notation: we fix K a field and X/K
smooth algebraic variety (i.e. finite type sep K -scheme)

Subvarieties = closed irreducible and reduced subvarieties
"integral"

Recall that smooth $\Rightarrow$ integral.

Rmk: Even though it is possible to grade everything by dimension, the good properties are there when one grades by codimension.
Since for X smooth the two approaches are equivalent, I stick with the second

Def An algebraic cycle of codim d is an element of $Z^d(X) := \bigoplus_V \mathbb{Z}[V]$ where V runs over all the irreducible subvarieties of codimension d in X (they have dim $n-d$ by smoothness).
not any subscheme.

in general, X/K irred and $Y \hookrightarrow X$ closed, then
 $\dim Y + \text{codim}_X Y = \dim X$. See GW PROP 5.30(2).

Def A codim d -cycle α on X is rationally equivalent to zero ($\alpha \sim_{\text{rat}} 0$) if there are finitely many subvar W_i of X of codimension $d-1$ and $r_i \in \mathbb{R}(W_i)^\times$
Since smooth $\Rightarrow$ irreducible, the claim follows.

[where $R(W_i) := \mathcal{O}_{W_i, \eta}$, η the generic point of W_i
 This agrees with the usual def of rational function because W_i is integral]

such that $\alpha = \sum [\text{div}(r_i)]$.

Recall: $\text{div}(r_i) := \sum_{V \in W_i \text{ codim } 1 \text{ subvar}} \text{ord}_V(r_i) [V]$

Example?

where, if $r_i \in \mathcal{O}_{W_i, V}$, one defines

$$\text{ord}_V(r_i) = \text{length}_{\mathcal{O}_{W_i, V}} (\mathcal{O}_{W_i, V} / (r_i))$$

In general $r_i = f/g$ for $f, g \in \mathcal{O}_{W_i, V}$ and one defines
 $\text{ord}(f/g) := \text{ord}_V(f) - \text{ord}_V(g)$ (and checks that it is well def)

$\mathcal{O}_{W_i, V} :=$
 local ring at
 generic point
 of V

One defines $CH^d(X) = Z^d(X) / \sim_{\text{rat}}$
 and $CH^*(X) := \bigoplus_{i=0}^n CH^i(X)$.

max length of a chain
 of submodules with
 simple quotients.

Operations on $CH^d(X)$:

Let $f: X \rightarrow Y$ proper

Let $V \subseteq X$ closed, then $f(V)$ is closed as well.

If $\dim(f(V)) = \dim(V)$, then

$R(V)/R(f(V))$ is a finite extension.

Def: $f_*([V]) = \begin{cases} [R(V):R(f(V))][f(V)] & \text{if } \dim V = \dim f(V) \\ 0 & \text{otherwise} \end{cases}$

$\leadsto$ extend linearly to

$$Z^d(X) \longrightarrow Z^{\dim Y - \dim(X) + d}(Y)$$

Multiplicativity of degrees of field extensions

$\Rightarrow f: X \rightarrow Y \quad g: Y \rightarrow Z$ proper, then

$$g_* \circ f_* = (g \circ f)_*$$

THM: If $X \xrightarrow{f} Y$ is a proper morphism of smooth K -varieties, and $\alpha \in Z^d(X)$ is

$\sim_{\text{rat}} 0$, then $f_* \alpha \sim_{\text{rat}} 0$

Prm K (8): The theorem follows by an explicit calculation: if $f: Z \rightarrow Y$ is proper surj morphism of varieties and $r \in R(Z)^\times$ then:

$$f_*[\text{div}(r)] = \begin{cases} 0 & \text{if } \dim(Y) < \dim(Z) \\ [\text{div}(N(r))] & \text{if } \dim(Y) = \dim(Z) \end{cases}$$

N is the determinant of mult by r on $R(Z)$.

Corollary: $f_*: Z^d(X) \rightarrow Z^{\dim(Y) - \dim(X) + d}(Y)$

induces $f_*: CH^d(X) \rightarrow CH^{\dim(Y) - \dim(X) + d}(Y)$.

(or just $f_*: CH^*(X) \rightarrow CH^*(Y)$, but one has to remember that it does not preserve the grading).

§ A different def of rational equivalence

Def (Cycle associated to a scheme)

Here I need the fund. cycle of a scheme.

Let X_k be a scheme of finite type,

let $D_1, \dots, D_r$ be its irred components.

The local rings $\mathcal{O}_{X, D_i}$ are Artinian, let m_i their length as $\mathcal{O}_{X, D_i}$ modules.

then $[X] = \sum_{i=1}^r m_i [D_i]$.

this is called the multiplicity of D_i in X .

Def (Variant of def 2)

$\alpha \in Z^d(X)$ is $\sim 0 \iff$ there are subvarieties

$V_1, \dots, V_t$ of $X \times \mathbb{P}^1$ of dimension $n-d+1$.

such that $f_i: V_i \hookrightarrow X \times \mathbb{P}^1 \xrightarrow{p} \mathbb{P}^1$ is dominant

and $\alpha = \sum_{i=1}^t \left(p_* [f_i^{-1}(0)] - p_* [f_i^{-1}(\infty)] \right)$.

$\nearrow$ cycle associated to the scheme $f^{-1}(0)$

Proof:

Suppose $\alpha \sim 0 \rightsquigarrow \exists W \subseteq X \quad r \in R(W)^X \quad \alpha = [\text{div}(r)]$.

Then $\exists U \subseteq W$ open s.t. $r|_U \in \mathcal{O}_U(U)$.

let V be the closure of $\text{graph}(r|_U)$ and consider

$$V \xrightarrow{w} W \times \mathbb{P}^1 \xrightarrow{p} \mathbb{P}^1$$

st $p: V \rightarrow W$ is surj and birational.

Then $[\text{div}(v)] = P_* [\text{div}(f)] = P_* ([f^{-1}(0)] - [f^{-1}(\infty)])$.
 From $\text{Rmk}(\Theta)$
 because $P|_V$ induces iso between $R(W)$ and $R(V)$.

this is a general computation that one can achieve by comparing degrees. (See end)

Rmk: Suppose that X_k is proper and $\alpha \sim_{\text{rat}} 0$.

Denote with $\pi: X \rightarrow \text{Spec}(k)$ the structure morph.

Then $\pi_*(\alpha) = \sum n_p [R(P):K] = 0$. This is what is usually called the degree.
 strictly speaking $\bullet [\text{Spec}(k)]$.

Def: $f: X \rightarrow Y$ flat of constant relative dim.

then for any subvar $V \in Y$ one defines

$$\begin{array}{ccc} Z^d(Y) & \longrightarrow & Z^d(X) \\ V & \longmapsto & [f^{-1}(V)] \end{array}$$

here one uses flatness

Again one has compatibility $g^* f^* = (fg)^*$.

and moreover f^* preserves rational equivalence, so one gets

$$f^*: CH^d(Y) \longrightarrow CH^d(X)$$

The ring structure on $CH^*(X)$ (a.k.a intersection product)

Although there is an explicit description of intersection product, I will adopt the following axiomatic definition:

Def / thm: (EH, Thm 1.5)

If X is smooth quasi-projective, then there is a unique product structure on $CH^*(X)$ s.t

if A, B subvar are generically transverse

i.e. if they meet transversally in a dense open of each component of $A \cap B$ (EH say at a general point)

Meeting transv. at $x \in X := A, B, X$ are smooth and

$$T_x X = T_x A + T_x B$$

Prmk being transverse is an open condition

then $[A] \cdot [B] = [A \cap B]$

↑
the class def as above.

This makes $CH^*(X)$ into an associative, commutative graded ring.

Moreover, the projection formula holds:

$$f_+ (f^* \alpha \cdot \beta) = \alpha \cdot f_+ \beta \quad (\text{EH THM 1.23})$$

The definition is based on the following moving lemma:

THM (Moving lemma) (EH: Thm 1.6)

X as above

1) for every $\alpha, \beta \in \mathbb{Z}^*(X)$ $\alpha = [A], \beta = [B]$
for A, B generically transverse.

2) $[A \cap B]$ is indep of the choice of A, B .

Formulas for computing $[A] \cdot [B]$:

THM (EH: Thm 1.26)

X smooth, A, B subvar s.t for every component C of $A \cap B$ one has

$$\text{codim}_X(C) = \text{codim}_X(A) + \text{codim}_X(B), \quad \text{then}$$

we call this: dimensionally transverse

$$[A] \cdot [B] = \sum_C m(C; A, B) [C].$$

and $m(C; A, B)$ (called intersection multiplicity at C)

is equal to 1 $\iff A, B$ intersect transversally at a general point of C

PROP (EH 1.14) X smooth.

For X_1, X_2 closed subvar of X , one has
(Mayer-Vietoris)

$$CH(X_1 \cap X_2) \rightarrow CH(X_1) \oplus CH(X_2) \rightarrow CH(X_1 \cup X_2) \rightarrow 0$$

(Excision)

$Z \subset X$ closed subvar, and $U = X \setminus Z \subset X$
then one has

$$CH(Z) \xrightarrow{i_*} CH(X) \xrightarrow{j^*} CH(U) \rightarrow 0$$

Computation of Chow rings with stratifications:

Def A stratification of X is a finite collection of irreducible, locally closed subschemes U_i s.t. $X = \bigsqcup U_i$ and $\overline{U_j}$ is a union of strata.
 $\uparrow$ called strata

$\overline{U_i}$ are called the closed strata

A stratification is quasi-affine if $U_i \cong$ open in some A^k .

PROP (EH 1.17)

If X has a quasi-affine stratification,
then $CH(X)$ is given by the classes
of the closed strata.

Proof Use excision and the fact that
for U quasi-affine $CH(U) = [U] \cdot \mathbb{Z}$

(EH 1.15)

PROP (EH THM 2.1)

$$CH(\mathbb{P}^n) \cong \mathbb{Z}[\beta]/(\beta^{n+1}).$$

where $\beta = [H]$ for H any hyperplane.

Proof:

Use the stratification

$$\{0\} \subset \mathbb{P}^1 \subset \cdots \subset \underbrace{\mathbb{P}^{n-1}}_H \subset \mathbb{P}^n$$

Thus the Chow ring is gen by closed strata

$$\text{On the other hand } CH^n(\mathbb{P}^n) = \mathbb{Z}[0]$$

Any $(n-k)$ -plane intersects a general k -plane H in one (reduced) point

$$\Rightarrow CH^k(\mathbb{P}^n) \xrightarrow{\cdot [H]} CH^n(\mathbb{P}^n) \cong \mathbb{Z}$$

intersection
with the class
of H

is surj. Since $CH^k(\mathbb{P}^n)$ is gen by a unique element, it has to be freely generated by that.

Finally, since a $(n-k)$ -plane L is

the intersection of k hyperplanes
(that can be chosen so that the intersection is generically transverse)

$$[L] = \beta^k.$$

Example (that $[A] \cdot [B] \neq [A \cap B]$ in general)

Take $X = \mathbb{P}^4 = \text{Proj}(K[x_0, \dots, x_4])$.

$$V_1 := V_+(x_0, x_1) \quad V_2 := V_+(x_2, x_3)$$

$$A := V_1 \cup V_2 = V_+(x_0 x_2, x_0 x_3, x_1 x_2, x_1 x_3)$$

$$B = V_+(x_0 - x_2, x_2 - x_3) \quad B' = V_+(x_3 - x_4, x_2 - x_3)$$

Notice $V_1 \cap B = V_1 \cap V_2 = V_2 \cap B = 0 := [0:0:0:0:1]$.

To compute $[A] \cdot [B]$ first observe that $[B] = [B']$

e.g. take $W = V_+(x_1 - x_3)$, then $[B] - [B'] = [\text{div}(\frac{x_0 - x_2}{x_3 - x_4})]$ or it follows from the computation before.

$$B' \cap V_1 = V_+(x_0, x_1, x_3, x_4) = [0:0:1:0:0] =: P_1'$$

$$B' \cap V_2 = V_+(x_2, x_3, x_1, x_4) = [1:0:0:0:0] =: P_2'$$

B' meets $A = V_1 \cup V_2$ ^{generically} transversally and thus $[A] \cdot [B] = [A] \cdot [B'] = [P_1'] + [P_2'] = 2[0]$

We show that $[A] \cdot [B] \neq [A \cap B]$.

Indeed $[A \cap B]$ has the unique irreducible component corresponding to the point 0

We compute $m(0, A \cap B)$: Take affine coord $x_i := \frac{x_i}{x_4}$

$$\mathcal{O}_{A \cap B, 0} := \frac{K[x_0, \dots, x_3]_{m_1}}{m_1} \cong \frac{K[x_0, x_1]_{(x_0, x_1)}}{(x_0^2, x_1^2)} \cong k[x_0] \oplus k[x_0] \oplus k[x_1]$$

$$I(A) + I(B) \rightarrow (x_0 x_2, x_0 x_3, x_1 x_2, x_1 x_3, x_0 - x_2, x_2 - x_3) \quad (x_0^2, x_1^2, x_3^2) \quad \text{length} = 3.$$

so $[A \cap B] = 3[0]$. Thus $[A] \cdot [B] \neq [A \cap B]$ because they have different degrees.

One can explicitly describe the numbers $m(C; A, B)$ using Serre's Tor formula:

THM (EH Thm 2.7)

Notation as in the last thm:

$$m(C; A, B) = \sum_{i=0}^{\dim X} (-1)^i \text{length}_{A \cap B, C} (\text{Tor}_i^{O_{X,C}}(O_{A,C}, O_{B,C}))$$

Remark:

The term for $i=0$ on the right is just:

$$\text{length}_{A \cap B, C} (O_{X,C} / I_A + I_B) \text{ which is exactly}$$

the multiplicity of C in $A \cap B$.

One can use that $\text{Tor}_1^R(R/I, R/J) \cong \frac{I \cap J}{I \cdot J}$ and check that in an example $(I=I_A, J=I_B)$ the length is 1.

§ Line bundles and Chern classes (after GW)

If X is smooth (in fact regular is enough)

then this is the isom that everyone has seen [FU, Prop 1.6].

$$\begin{array}{ccc} \text{Cart}(X) & \xrightarrow{\sim} & \text{Weil}(X) \\ \parallel & \downarrow \text{Princ}(\text{Cart}) & \downarrow \text{Princ}(\text{Weil}) \\ \sim_{\mathbb{Q}} \text{Pic}(X) & & = \text{CH}^1(X) \end{array}$$

those that come from $T(X, R(X)^*)$

because X is integral: GW Prop 11.23.

$$\text{where } \frac{\text{Div}(X)}{\text{PDiv}(X)} \xrightarrow{\sim} \frac{\text{Weil}(X)}{\text{PWeil}(X)} \text{ is induced by cyc:}$$

Let $D \in \text{Div}(X)$ (Cartier)

$\forall C \in X^1$ let f^{be} the germ of f_i at η_C
for i s.t. $\eta_C \in U_i$.

$\text{ord}_C(D) := \text{ord}_{\mathcal{O}_{X,C}}(f)$ is indep of choice of i .

$$D \xrightarrow{\text{cyc}} \sum_{C \in X^1} \text{ord}_C(D) [C] =: [D].$$

(finite sum because if $C \notin \text{Supp}(D)$, then $\text{ord}_C(D) = 0$)

$$\text{Supp}(D) = \{x \in X \mid (f_i)_x \notin \mathcal{O}_{X,x}^\times\}.$$

Smoothness: suppose that X is smooth curve,

then every codim 1 subvar is locally
generated by a single function

(smooth $\Rightarrow$ locally factorial)

and thus it determines a Cartier divisor

Step 2: Chern classes in a special case

Prmk: we remark that a line bundle is ^(e.g. because of the iso above) trivial $\Leftrightarrow c_1(L) = 0$ and one would like to have higher analogues of classes c_1 .

One can regard c_1 as a measure of non triviality.

A bundle $\mathcal{E}$ of rank r is trivial $(\Rightarrow)$ it has r everywhere independent global sections.

Take $\tau_0, \dots, \tau_{r-1}$ rational sections of $\mathcal{E}$,

one can construct $\tau_0 \wedge \dots \wedge \tau_{r-1} \in \wedge^r \mathcal{E}$

which is a line bundle and one

defines $c_1(\mathcal{E}) := c_1(\wedge^r \mathcal{E})$

and we will have again $c_1(\mathcal{E}) = 0 \Leftrightarrow \mathcal{E}$ trivial

THM (EH, thm 5.3)

There is a unique way of assigning to each vector bundle $\mathcal{E}$ on a smooth quasi-proj variety X a class

$$c(\mathcal{E}) = 1 + c_1(\mathcal{E}) + \dots \in CH^*(X) \text{ s.t.}$$

• If $\text{rank } \mathcal{E} = 1$ then $c(\mathcal{E}) = 1 + c_1(\mathcal{E})$

• If $\tau_0, \dots, \tau_{r-i}$ are global sections such that $V = V(\tau_0 \wedge \dots \wedge \tau_{r-i})$ has codim i closed subscheme of X where their wedge is zero

then $c_i(\mathcal{E}) = [V] \in CH^i(X)$.

(this holds if $\mathcal{E}$ is generated by its global sections)
see FH Lemma 5.2

• If $0 \rightarrow \mathcal{E} \rightarrow \mathcal{F} \rightarrow \mathcal{G} \rightarrow 0$ short exact,

then $c(\mathcal{F}) = c(\mathcal{E}) \cdot c(\mathcal{G})$ (Whitney's formula)

• If $\varphi: Y \rightarrow X$ is a morph of smooth $\mathbb{A}^1$ -proj var

$$\varphi^*(c(\mathcal{E})) = c(\varphi^*(\mathcal{E}))$$

• (FH Example 5.15)

$$c_1(\mathcal{E}) = c_1(\det \mathcal{E}).$$

Step 3: Chem classes in the general case:

The general definition uses projective bundles and the projective bundle formula:

Def: Let $\mathcal{E}$ be a vector bundle of rank r on a variety X .

The projectivization of $\mathcal{E}$ is the X -scheme

$$\pi_{\mathcal{E}}: \mathbb{P}\mathcal{E} := \left(\text{Proj}(\text{Sym } \mathcal{E}^{\vee}) \right) \rightarrow X.$$

whose closed points are hyperplanes in $\mathcal{E}_x^{\vee}$
i.e. lines in $\mathcal{E}_x$, for $x \in X$.

$\pi: \mathbb{P}\mathcal{E} \rightarrow X$ comes with the tautological line bundle on $\mathbb{P}\mathcal{E}$

$\mathcal{O}_{\mathbb{P}\mathcal{E}}(-1)$ given by sheafifying the graded $\text{Sym } \mathcal{E}^{\vee}$ -mod obtained by shifting the grading by -1 .

$$\mathcal{O}_{\mathbb{P}\mathcal{E}}(1) := \mathcal{O}_{\mathbb{P}\mathcal{E}}(-1)^{\vee}.$$

THM (Projective bundle formula). (EH Thm 59)

Let X be smooth, $\pi: \mathbb{P}\mathcal{E} \rightarrow X$

$$x := c_1(\mathcal{O}_{\mathbb{P}\mathcal{E}}(1)) \in CH^1(\mathbb{P}\mathcal{E}).$$

where $\mathcal{E}$ is a vector bundle of rank r .

Then x satisfies a unique monic polynomial of degree r with coeff in $\pi^*(CH(X))$

$$f(y) := y^r + \pi^* c_1(\mathcal{E}) y^{r-1} + \dots + \pi^* c_r^+(\mathcal{E})$$

with $c_i(\mathcal{E}) \in CH^i(X)$. Moreover π^* is inj.

Def

The elements c_i are called i -th Chern class. One defines also $c_j(\mathcal{E}) = 0$ if $j \geq r+1$

Remark: More is true:

$$CH^*(\mathbb{P}\mathcal{E}) \cong \pi^*(CH^*(X)[y]/f(y))$$

One defines the total Chern class of $\mathcal{E}$ as

$$c(\mathcal{E}) = 1 + \sum_{i=1}^r c_i(\mathcal{E}) \in CH^*(X)$$

Whitney's formula allows to upgrade c to a group homom:

$$c : (K_0(X), +) \longrightarrow (CH^*(X)^{\times}, \cdot_{\text{intersection}})$$

$$\mathcal{E} \longmapsto c(\mathcal{E})$$

It does not induce a ring hom structure
(clearly: it sends $0 \mapsto 1$)

Next time: we will see how to
modify c to get a
ring homom

Stuff that I didn't include and
computation for equivalent def of rational eq.:

Frac if X is integral $\mathcal{R}(X) = \text{Frac}(\mathcal{O}_X(X))$.

Let $f \in \mathcal{R}(X)^\times \rightsquigarrow \tilde{f}: X \rightarrow \mathbb{P}^1$
rational morphism.

$$f = f_1/f_2 \quad f_1, f_2 \in \mathcal{O}_X(X) = \text{Hom}(X, \mathbb{A}^1) \subset \text{Hom}(X, \mathbb{P}^1)$$

Want: $[\text{div}(f)] = [\tilde{f}^{-1}(0)] - [\tilde{f}^{-1}(\infty)]$

$$[\text{div}(f)] = [\text{div}(f_1)] - [\text{div}(f_2)].$$

So I have to show that

$$[\text{div}(f_1)] = [\tilde{f}_1^{-1}(0)]$$

|| def

$$\sum_{C \in X}$$

$$\text{ord}_C(f_1)[C]$$

$$l_{\mathcal{O}_{X,C}}(\mathcal{O}_{X,C}/(f_1))$$

ord 1 (irreducible)

(if $f_1 \neq 0$ on C , then this length is 0.)

$$\sum_{C \text{ irred comp of } \tilde{f}_1^{-1}(0)} \text{ord}_C(f_1)[C].$$

$$\leftarrow =: l_{\mathcal{O}_{X,C}}(\mathcal{O}_{X,C}/(f_1))$$

$$\sum_{\substack{C \text{ irred comp of } \tilde{f}_1^{-1}(0)}} l_{\mathcal{O}_{X,C}}(\mathcal{O}_{X,C}/(f_1))$$

$$l_{\mathcal{O}_{\tilde{f}_1^{-1}(0),C}}(\mathcal{O}_{\tilde{f}_1^{-1}(0),C})[C]$$

$$l_{\mathcal{O}_{X,C}}(\mathcal{O}_{X,C}/(f_1))$$

because $\tilde{f}_1 = 0$

in $\mathcal{O}_{\tilde{f}_1^{-1}(0),C}$.

§ Splitting principle.

Lemma 5.12 of EH:

Let X be a smooth variety and $\mathcal{E}$ a vector bundle of rank r on X .

There exists $\varphi: Y \rightarrow X$ of smooth var. s.t.

a) $\varphi^*: CH(X) \rightarrow CH(Y)$ is inj.

b) $\varphi^* \mathcal{E}$ has a filtration

$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \dots \subset \mathcal{E}_r = \varphi^* \mathcal{E}$ s.t. each subquotient is a line bundle.

Example (EH § 5.6.1)

For $\mathbb{P}^n$, the class $c_1(\mathcal{O}_{\mathbb{P}^n}(1)) \in CH^1(\mathbb{P}^n)$ is the hyperplane class.

Proof

By construction, the equation of any hyperplane is the zero locus of a section of $\mathcal{O}_{\mathbb{P}^n}(1)$.

The claim follows by the computational description of Chern classes (or in fact by the very definition).