Waldhausen $S_\bullet$-construction
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Abstract

In this talk, we introduce the notion of Waldhausen $S_\bullet$-construction and the corresponding algebraic K-theory. First, we introduce the idea of Waldhausen Categories and the simplicial categories $S_n\mathcal{C}$, which lead to the definition of Waldhausen K-theory. We also show that one can realize the Waldhausen K-theory as an Ω-spectrum. Finally, we also briefly sketch the idea of non-connective K-theory.

Contents

1 Motivation

In the previous lectures, we have encountered various notions of defining higher K-theory groups: the plus construction, the $S^{-1}S$ construction, and the Q-construction. These constructions helped us reformulate the higher K-theory groups as homotopy of groups of a topological space. In particular, we have defined K-theory for exact categories, which are additive categories with additional structure of ”exact sequences”.

But what about K-theory for non-exact categories? Such categories arise from a topological nature. A prominent example of this is the category of CW complexes.

Example 1.1. [5, Topological Example 9.1.4] Let $\mathcal{R}$ be the category of based CW complexes with countably many cells (we fix a cardinal to bound the number of cells) and morphisms being cellular maps. Let $\mathcal{R}_f$ be the subcategory of finite-based CW complexes.

The following categories are not additive, as we cannot endow the space of continuous maps with an abelian structure (there is no usual way to ”add” continuous maps of topological spaces). As finite CW complexes are built by spheres S^n and every sphere is also glued by contractible spaces D^n and a sphere of lower dimension S^{n-1}. Thus, one would expect the $K_0(\mathcal{R}_f)$ to be

$$\mathbf{Z}[[S^0]]. \tag{1}$$

Intuitively, we used cellular inclusions as "admissible monomorphisms" and weak equivalences of topological spaces to obtain such a result. Waldhausen categories formalize this idea in abstract category theory.

There is another motivation for why Waldhausen K-theory. It encodes negative K-theory. In the context of rings, Bass has defined negative K-theory, which vanishes when the ring is regular. If the ring is not regular, then the K-theory groups are negative. The Q-construction gives us a topological space, and this does not allow us to encode negative K-groups. The language of spectra will enable us to encode negative K-groups. It turns out that Waldhausen non-connective K-theory is defined as a colimit of connective spectra (i.e., spectra with trivial negative homotopy groups) that are non-connective. Such techniques help us determine the K-theory of general schemes, a generalization of Bass's K-theory.

Remark 1.2. The group $K_{-1}R$ is the cokernel of the map

$$K_0(R[t]) \oplus K_0(R[t^{-1}]) \xrightarrow{\pm} K_0(R[t,t^{-1}] \tag{2}$$

For R a ring the above map is surjective as $K_0(R[t_1,t_2,\cdots,t_n] = K_0(R)$.

Inductively, one can define negative K-groups for a ring R using the above equation (see [5, Page 229] for details).

2 Waldhausen categories

Definition 2.1. [5, Definition 9.1] Let $\mathcal{C}$ be a category equipped with a subcategory co($\mathcal{C}$) of morphisms in a category $\mathcal{C}$, called "cofibrations" (indicated by $\rightarrowtail$). The pair $(\mathcal{C}, \text{co}(\mathcal{C}))$ is called a *category with cofibrations* if the following axioms are satisfied:

(W0) Every isomorphism in $\mathcal{C}$ is a cofibration.

(W1) There is a distinguished zero object "0" in $\mathcal{C}$ and the unique map $0 \rightarrowtail A$ is a cofibration for every $A \in \mathcal{C}$.

(W2) If $A \rightarrowtail B$ is a cofibration and $A \to C$ is any morphism in $\mathcal{C}$, then the pushout $B \cup_A C$ exists in $\mathcal{C}$, and morever the map $C \rightarrowtail B \cup_A C$ is a cofibration.

$$\begin{array}{ccc} A & \rightarrowtail & B \\ \downarrow & & \downarrow \\ C & \rightarrowtail & B \cup_A C \end{array} \tag{3}$$

Remark 2.2. The above definitions allow us to make sense of these two constructions:

1. Coproducts of two objects exist in $\mathcal{C}$.
2. Let $i : A \rightarrowtail B$ be a cofibration in $\mathcal{C}$. By $W2$, the following pushout diagram exists :

$$\begin{array}{ccc} A & \overset{i}{\rightarrowtail} & B \\ \downarrow & & \downarrow \\ 0 & \rightarrowtail & B \cup_A 0 =: B/A \end{array} \tag{4}$$

We see that B/A is the cokernel of the map i. We call the sequence $A \rightarrowtail B \twoheadrightarrow B/A$ a cofibration sequence.

Definition 2.3. [5, Definition 9.1.1] A *Waldhausen Category* $\mathcal{C}$ is a category with cofibrations together with a family $w(\mathcal{C})$ of morphisms in $\mathcal{C}$ called "weak equivalences" (indicated by $\xrightarrow{\sim}$). Isomorphisms and compositions of weak equivalences are weak equivalences (so we may regard $w(\mathcal{C})$ as a subcategory of $\mathcal{C}$). In addition, the following "Glueing Axiom" must be satisfied.

W3 *Glueing for weak equivalences*: For every commutative diagram of the form

$$\begin{array}{ccccc} C & \longleftarrow & A & \rightarrowtail & B \\ \downarrow\sim & & \downarrow\sim & & \downarrow\sim \\ C' & \longleftarrow & A' & \rightarrowtail & B' \end{array} \tag{5}$$

(in which the vertical maps are weak equivalences and the two right horizontal maps are cofibrations), the induced map

$$B \cup_A C \to B' \cup_{A'} C' \tag{6}$$

is a weak equivalence.

Example 2.4. 1. The category $\mathcal{R}$ is a Waldhausen category by cofibrations as cellular inclusions and weak equivalences as weak equivalences of topological spaces.

2. Any exact category is a Waldhausen category where cofibrations are admissible monomorphisms and weak equivalences are isomorphisms.

3. Let $\mathcal{A}$ be an abelian category, then $\mathrm{Ch}(\mathcal{A})$, the category of Chain complexes, admits a Waldhausen structure by cofibrations as degreewise monic and weak equivalences as quasi-isomorphisms

Definition 2.5. A functor $f : \mathcal{A} \to \mathcal{C}$ between two Waldhausen categories is called an *exact functor* if it preserves all the relevant structure, zero, cofibrations, weak equivalences, and the pushouts along a cofibration.

3 K-theory of Waldhausen Categories.

We first define the K_0 of a Waldhausen category.

Definition 3.1. [5, Definition 9.1.2] Let $\mathcal{C}$ be a Waldhausen category. $K_0(\mathcal{C})$ is the abelian group presented as having one generator $[C]$ for every $C \in \mathcal{C}$ subject to the relations

1. $[C] = [C']$ if there is a weak equivalence $C \xrightarrow{\sim} C'$.

2. $[C] = [B] + [C/B]$ for every cofibration sequence $B \rightarrowtail C \twoheadrightarrow C/B$.

Remark 3.2. For the definition to make sense, the set of weak equivalences should be assumed as a set. This is assumed intrinsically from now on. In particular, one chooses a set of weak equivalences. We also see that $[C] = 0$ whenever $0 \sim C$.

Proposition 3.3. *[5, Proposition 9.1.5]* $K_0\mathcal{R}_f \cong \mathbf{Z}$.

Proof. As explained in the motivation, every object in $\mathcal{R}_f$ is generated by $[S^0]$. Also, the reduced Euler characteristic map gives a surjection from $K_0(\mathcal{R}_f) \to \mathbf{Z}$. Hence, this is an isomorphism. □

To define the Waldhausen $wS_\bullet$ construction, we need to introduce additional terminology.

Definition 3.4. [5, Definition 8.3] If $\mathcal{C}$ isa category with cofibrations, let $S_n\mathcal{C}$ be the category whose objects $A_\bullet$ are sequences of n-cofibrations in $\mathcal{C}$:

$$A_\bullet : 0 = A_0 \rightarrowtail A_1 \rightarrowtail A_2 \rightarrowtail \cdots \rightarrowtail A_n \tag{7}$$

together with a choicse of every subquotient $A_{ij} := A_j/A_i(0 < i < j \leq n)$. These choices are to be compatible in the sense that there is a commutative diagram:

$$\begin{array}{ccccccc}
 & & & & & & A_{n-1,n} \\
 & & & & & & \uparrow \\
 & & & & & & \vdots \\
 & & & & & & \uparrow \\
 & & & & A_{23} & \rightarrowtail \cdots \rightarrowtail & A_{2n} \\
 & & & & \uparrow & & \uparrow \\
 & & A_{12} & \rightarrowtail & A_{13} & \rightarrowtail \cdots \rightarrowtail & A_{1n} \\
 & & \uparrow & & \uparrow & & \uparrow \\
A_1 & \rightarrowtail & A_2 & \rightarrowtail & A_3 & \rightarrowtail \cdots \rightarrowtail & A_n
\end{array} \tag{8}$$

Morphisms in S_nC are natural transformations of sequences.

Definition 3.5. A weak equivalence in S_nC is a map $A_\bullet \to B_\bullet$ such that for i the map $A_i \to B_i$ is a weak equivalence. A map $A_\bullet \to B_\bullet$ is a cofibration when for every $0 \leq i < j < k \leq n$, the map of cofibration sequences

$$\begin{array}{ccccc}
A_{ij} & \rightarrowtail & A_{ik} & \twoheadrightarrow & A_{jk} \\
\downarrow & & \downarrow & & \downarrow \\
B_{ij} & \rightarrowtail & B_{ik} & \twoheadrightarrow & B_{jk}
\end{array} \tag{9}$$

where the vertical leftmost and rightmost arrows are cofibrations, as well as the map $B_{ij} \cup_{A_{ij}} A_{ik} \to B_{ik}$ is a cofibration.

The above definition gives a Waldhausen structure on the categories $S_n\mathcal{C}$.

Definition 3.6. [5, Definition 8.3.1] For each $n \geq 0$, we define the following maps:

1. Functors $\partial_i^n : S_n\mathcal{C} \to S_{n-1}\mathcal{C}$ is defined by the formula:

$$\partial_i^n(A_\bullet) := \begin{cases} 0 = A_{11} \rightarrowtail A_{12} \cdots \rightarrowtail A_{1n} & i = 0 \\ A_1 \rightarrowtail A_2 \rightarrowtail \cdots \widehat{A_i} \cdots A_n & 0 < i \leq n \end{cases} \tag{10}$$

   where $\hat{A}_i$ means omitting A_i in the equation.

2. Functors $s_i^n : S_n\mathcal{C} \to S_{n+1}\mathcal{C}$ for $0 \leq i \leq n$ is defined by the following formula:

$$s_i^n(A_\bullet) := A_1 \rightarrowtail A_2 \cdots A_i \xrightarrow{\text{id}} A_i \cdots \rightarrowtail A_n \tag{11}$$

Proposition 3.7 (Left as an exercise). *The functors ∂_i^n and s_i^n are exact and give rise to a simplicial Waldhausen category :*

$$S_\bullet\mathcal{C} : \Delta^{\text{op}} \to \text{Cat} \tag{12}$$

sending $[n] \mapsto S_nC$ and the face and degeneracy maps to ∂_i^n and s_i^n respectively.

Definition 3.8. Let $wS_n\mathcal{C}$ be the subcategory of $S_n\mathcal{C}$ where the maps between objects are weak equivalences. We define the topological space $|wS_\bullet\mathcal{C}|$ as

$$|wS_\bullet\mathcal{C}| := |N(wS_\bullet\mathcal{C})| \cong |\operatorname{diag}(N(wS_\bullet\mathcal{C}))|. \tag{13}$$

In other words, the topological space $|wS_\bullet\mathcal{C}|$ is the geometric realization of the bisimplicial set $N(wS_\bullet\mathcal{C})$.

Remark 3.9. For X a Kan complex where the 0-simplex is a point, i.e., a simplicial set, i.e., where every edge is an equivalence. Then one can define $\pi_1(X_\bullet) = \mathbf{Z}[X_1]/R$ where R is relation generated by

$$\{\partial_1^2[X] = \partial_0^2[X] + \partial_2^2[X] \mid [X] \in X_2\}. \tag{14}$$

Proposition 3.10. *[5, Proposition 8.4] If $\mathcal{C}$ is a Waldhausen category, then $\pi_1|wS_\bullet\mathcal{C}| \cong K_0(\mathcal{C})$.*

Proof. Using the remark above, we see that $\mathbf{Z}[(NwS_1)_1]$ are maps $A \sim B$ of weak equivalences. Elements in $\mathbf{Z}(NwS_2)_2]$ are composable sequences of three objects which are cofibration sequences. The face maps are defined in such a way that gives the relation $[B] = [A] + [B/A]$ for a cofibration sequence $A \rightarrowtail B \twoheadrightarrow B/A$. □

Motivated by the above proposition, we are ready to define the algebraic K-theory

Definition 3.11. [5, Definition 8.5] If $\mathcal{C}$ is a small Waldhausen category. Its *algebraic K-theory space* $K(\mathcal{C}) = K(\mathcal{C}, w)$ is the loop space

$$K(\mathcal{C}) := \Omega|wS_\bullet\mathcal{C}|. \tag{15}$$

The K-groups of $\mathcal{C}$ are defined to be its homotopy groups:

$$K_n(\mathcal{C}) = \pi_n K(\mathcal{C}) = \pi_{n+1}|wS_\bullet\mathcal{C}|. \tag{16}$$

4 K-theory as Ω-spectrum.

To make sense of the Ω-spectrum structure of K-theory, we need some additional terminology.

Notation 4.1. If $f : \mathcal{B} \to \mathcal{C}$ os an exact functor, we can define the Relative K-theory space which admits the following properties:

1. Let $S_n f := S_n\mathcal{B} \times_{S_n\mathcal{C}} S_{n+1}\mathcal{C}$ where the map from $S_{n+1}\mathcal{C} \to S_n\mathcal{C}$ is the face map ∂_0^n.
2. $\mathcal{C}$ sits as a subcategory of $S_n f$ by $c \to (0, c = c \cdots = c)$.
3. The collections $S_n f$ naturally form a simplicial Waldhausen category $S_\bullet f$.
4. We have the sequence of exact maps of simplicial Waldhausen categories:

$$\mathcal{C} \to S_\bullet f \to S_\bullet\mathcal{B} \tag{17}$$

   where $\mathcal{C}$ is the constant simplicial category.

5. Applying $S_\bullet$ to the above sequence and iterating it on the right side, we get a long sequence of exact functors of bisimplicial Waldhausen categories:

$$wS_\bullet\mathcal{C} \to wS_\bullet(S_\bullet f) \to wS_\bullet(S_\bullet\mathcal{B}) \tag{18}$$

6. By [5, V, Proposition 1.7], the above sequence, upon taking geometric realizations, gives us a homotopy fibration sequence.
7. We can define $K(f) = \Omega^2|wS_\bullet(S_\bullet f)|$.

8. The homotopy groups fit into an exact sequence ending with :

$$\cdots K_0(\mathcal{B}) \to K_0(\mathcal{C}) \to K_{-1}(f) \to 0. \tag{19}$$

An interesting fact from the above notation is the following:

Lemma 4.2. *[5, Lemma 8.5.4] If $f : \mathcal{C} \to \mathcal{C}$ is the identity, then $|\omega S_\bullet(S_\bullet f)|$ is contractible.*

Proof. For f to be the identity, $wS_\bullet f$ recovers the classical construction of simplicial path space associated to a simplicial set $PwS_\bullet\mathcal{C}$. The simplicial path space associated to a simplicial set X (denoted by $PX_\bullet$) is the simplicial set

$$PX_n := X_n \times_{X_n,\partial_0^n} X_{n+1}. \tag{20}$$

One of the major classical properties of this simplicial path space is its homotopy equivalence to the constant simplicial set X_0. The brief sketch of this argument is as follows:

1. We have the projection map $p : PX_\bullet \to X_\bullet$ just projection on each degree.
2. We have the inclusion map $i : X_0 \to PX_\bullet$ induced by degeneracy maps on each degree.
3. The idea to show it is a homotopy equivalence is to show that the composition of these two maps in both directions is homotopic to the identity.
4. By definition, it follows that $p \circ i = \mathrm{id}_{X_0}$.
5. The hard part is to verify that there exists a homotopy $i \circ p \sim \mathrm{id}_{PX_\bullet}$. This needs some work. Please refer to [6, Section 8.3.1] for more details.

Thus, if X_0 is a point, then $PX_\bullet$ is contractible. By definition, it follows that in our case $S_\bullet f$ is a point, and hence $wS_0(S_0 f)$ is a point as well. Hence, we see that $\Omega S_\bullet(S_\bullet f)$ is contactible. □

Remark 4.3. Let us briefly recall the key players in spectra.

1. A spectrum is a sequence of pointed topological spaces $(E_n)_{n\in\mathbf{N}}$ with structure maps $\Sigma E_n \to E_{n+1}$. One can construct a category of spectra by choosing morphisms to be maps between such sequences that preserve the structure of spectra as expected. Let Spectra be the category of Spectra
2. To such a spectrum, we have suspension spectrum
$$\Sigma(E_\bullet)_n = E_{n+1}.$$
3. The loop space functor is defined as the right adjoint of the suspension functor
$$(\Omega E_\bullet)_n = \Omega E_n$$
4. We can define the stable homotopy groups of spectra $E_\bullet$ by the following formula:
$$\pi_k^s(E_\bullet) := \mathrm{colim}_{n\to\infty}\, \pi_{k+n}(E_n) \tag{21}$$
A spectrum is said to be *connective* if all its negative homotopy groups are zero.
5. A spectrum is said to be an Ω-*spectrum*, if the right adjoints to the structure maps $\Omega E_n \to E_{n+1}$ are equivalences.
6. There exists a model categorical structure on the category of spectra which says any spectrum can be "cofibrantly replaced" by an Ω-spectrum.

Corollary 4.4. *The topological space $|wS_\bullet\mathcal{C}|$ admits a structure of an Ω-spectrum.*

Proof. By Theorem 4.2 and using Eq. (19), we see that $\Omega|wS_\bullet(S_\bullet\mathcal{C})| \cong |wS_\bullet\mathcal{C}|$. This shows that the sequence of based point topological spaces (with 0 being the base point):

$$(\Omega|wS_\bullet\mathcal{C}|, |wS_\bullet\mathcal{C}|, |wS_\bullet(S_\bullet\mathcal{C})|, \cdots |wS^n_\bullet\mathcal{C}| \cdots) \tag{22}$$

fits into the definition of being an Ω-spectrum. □

Notation 4.5. The above Ω-spectrum is denoted by $\mathbf{K}\mathcal{C}$ and called as the *K-theory spectrum of* $\mathcal{C}$. An exact functor $f : \mathcal{B} \to \mathcal{C}$ between simplicial Waldhausen categories induces a map of Ω-spectra $f_* : \mathbf{K}\mathcal{B} \to \mathbf{K}\mathcal{C}$.

Lemma 4.6. *The Ω-spectrum $\mathbf{K}\mathcal{C}$ is connective.*

Proof. The proof follows from computation. Let k be a negative integer

$$\begin{aligned}
&\pi^s_k(\mathbf{K}\mathcal{C}) \\
&:= \mathrm{colim}_{n\to\infty}\, \pi_{n+k}(|wS^n_\bullet\mathcal{C}|) \\
&\cong \mathrm{colim}_{n\to\infty}\, \pi_0(\Omega^{n+k}|wS^n_\bullet\mathcal{C})|) \\
&\cong \mathrm{colim}_{n\to\infty}\, \pi_0|wS^{-k}\mathcal{C}| \ \ (\mathit{Theorem}\ 4.4) \\
&\cong 0 \ \ (\text{as } |wS^{-k}\mathcal{C}| \ \text{ is connected})
\end{aligned}$$

□

The theorem below shows that Waldhausen K-theory recovers Quillen K-theory.

Theorem 4.7. *[4, Section 1.9] Consider an exact category $\mathcal{A}$ as a Waldhausen category, there is a homotopy equivalence between the topological spaces $\Omega|wS_\bullet\mathcal{A}|$ and $BQ\mathcal{A}$.*

5 Non-connective K-theory

So far, we have seen how one obtains a connective K-theory spectrum from a Waldhausen category. However, to recover Bass's negative K-groups, we need to define a spectrum that need not be connected. This is done by defining Cone and Suspension functors. These ideas are due to Schlichting ([2]) and, in modern language, to Blumberg, Gepner, and Tabuada ([1]). We do need some technical assumption on the existence of cylinder functors ([5, Definition 8.8]) to make the definition precise. We do not recall the details, but we aim to give a general picture of how non-connective K-theory is defined.

Notation 5.1. Let $\mathcal{C}$ be a Waldhausen Category. Then we have an exact sequence of Waldhausen categories :

$$\mathcal{C} \to \mathcal{F}\mathcal{C} \to \Sigma\mathcal{C} \tag{23}$$

with the property that $\mathcal{F}\mathcal{C}$ admits countable direct sums, hence by Eilenberg Swindle, $K(\mathcal{F}\mathcal{C}) = 0$. The category $\mathcal{F}\mathcal{C}$ has objects which are sequences of cofibrations in $\mathcal{C}$ (which are not necessarily finite). The category $\mathcal{F}\mathcal{C}$ is called the Cone category, and $\Sigma(\mathcal{C})$ is the suspension category defined as the localization of $\mathcal{F}\mathcal{C}$ along the subcategory $\mathcal{C}$.

Remark 5.2.

1. Taking geometric realizations, we say that $\mathbf{K}(\mathcal{C}) \cong \Omega\mathbf{K}(\mathcal{F}\mathcal{C})$.
2. In the context of $\mathcal{C}$ as a stable ∞-category, then $\mathcal{F}(\mathcal{C}) = \mathrm{Ind}_\kappa(\mathcal{C})$ and $\Sigma\mathcal{C} := \mathrm{Ind}_\kappa(\mathcal{C})/\mathcal{C}$.

Definition 5.3. Let $\mathcal{C}$ be a Waldhausen category. The *non-connective K-theory* spectrum is defined as

$$\mathbb{K}(\mathcal{C}) := (|\Omega|wS_\bullet\mathcal{C}|, \Omega|wS_\bullet(\Sigma\mathcal{C})|, \cdots, \Omega|wS_\bullet(\Sigma^n\mathcal{C})|, \cdots). \tag{24}$$

In particular, for $n \geq 0$, the negative groups are defined as

$$\mathbb{K}_{-n}(\mathcal{C}) = K_0(\Sigma^n\mathcal{C}). \tag{25}$$

Remark 5.4.

1. In [2], Schlichting shows that such a definition recovers Bass's negative K-groups for non-regular rings.
2. Thomason-Trobaugh's K-theory ([3]) also uses non-connective K-theory to get proto-localization sequence in K-theory. For a scheme X, the K-theory of X is defined as the non-connective K-theory spectrum of the category of Perfect complexes on X.

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