Algebraic Geometry II Prof. Dr. Ulrich Görtz
SS 2026 Dr. Andreas Pieper

Problem sheet 6

Due date: June 2nd, 2026.

Problem 16 Let k be a field. Consider the homogeneous ideal $I = (x_0^2, x_0x_1, x_0x_2) \subset k[x_0, x_1, x_2]$. Show that $\mathcal{V}_+(I) \subset \mathbb{P}^2_k$ is reduced although I is not a radical ideal.

Problem 17(Cartier divisors) Let k be a field and $d, n \geqslant 1$ be integers.

(1) Let $F \in k[x_0, \dots, x_n]$ be a homogeneous polynomial of degree d. Show that

$$\operatorname{div}(F) = \left(D_+(x_i), \frac{F}{x_i^{\deg(F)}} \right)_i$$

is a Cartier divisor in $\mathbb{P}^n_k$ with support $\mathcal{V}_+(F)$.

(2) Now let $F_1, F_2 \in k[x_0, \dots, x_n]$ be two homogeneous polynomial of the same degree d. Denote $D_i := \operatorname{div}(F_i)$, $i = 1, 2$. Show that $\mathcal{O}_{\mathbb{P}^n_k}(D_1) \cong \mathcal{O}_{\mathbb{P}^n_k}(D_2)$.

Problem 18 (UFDs) Let R be a unique factorization domain. The goal of this exercise is to prove that $\operatorname{Pic}(R) = 0$.

(1) Let M be an invertible R-module. Prove that there exists an ideal $I \subset R$ and an isomorphism $M \xrightarrow{\sim} I$.
Hint: One way to proceed is to use that the natural map $M \otimes_R \operatorname{Hom}_R(M, R) \to R$, $m \otimes \varphi \mapsto \varphi(m)$ is an isomorphism. In particular $\operatorname{Hom}_R(M, R) \neq 0$.

(2) Now let $I \subset R$ be an ideal. Show that there exists a unique principal ideal (a) such that $I \subseteq (a)$ and (a) is minimal among the principal ideals containing I.
Hint: Since R is a UFD, you can use prime factorizations to define gcds.

(3) Assume now that $I \subset R$ is an ideal which is invertible as an R-module. Define a as in (2) and prove that under these assumptions $I = (a)$.
Hint: Equality of ideals can be checked locally.