Algebraic Geometry I Prof. Dr. Ulrich Görtz
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Problem sheet 7

Due date: Dec. 9, 2025.

Problem 23 Give an example of a topological space X, a surjective map $\mathscr{F} \to \mathscr{G}$ of sheaves on X and an open $U \subseteq X$ such that the map $\mathscr{F}(U) \to \mathscr{G}(U)$ is not surjective.

Hint: If you know complex analysis, you could give an example using sheaves of holomorphic functions and that one cannot define a logarithm on all of $\mathbb{C}^\times$. For an algebraic example consider $X = \mathbb{A}^2_k$, $\mathscr{F} = \mathscr{O}_X$ and $U = X \smallsetminus \{(0,0)\}$ as in Problem 21. For $\mathscr{G}$ you can take $i_*\mathscr{O}_Z$, where $Z = \operatorname{Spec}(k[x,y]/(y)) \cong \operatorname{Spec}(k[x])$ and $i\colon Z \to X$ is the map attached to the canonical projection $k[x,y] \to k[x,y]/(y)$.

Problem 24 Let X be a topological space, $Z \subset X$ a closed subspace, and $\mathscr{F}$ a sheaf of abelian groups on Z. Denote by $i : Z \to X$ the inclusion map. Show that for $x \in X$

$$(i_*\mathscr{F})_x = \begin{cases} \mathscr{F}_x & \text{if } x \in Z, \\ 0 & \text{otherwise.} \end{cases}$$

Problem 25 Let $F : \mathscr{C} \to \mathscr{D}$, $G : \mathscr{D} \to \mathscr{C}$ be two functors. We say that F is *left adjoint* to G (or equivalently, that G is *right adjoint* to F, or that F, G are an adjoint pair, sometimes written as $F \dashv G$), if there exists a collection of isomorphisms

$$\operatorname{Hom}_{\mathscr{D}}(F(X), Y) \cong \operatorname{Hom}_{\mathscr{C}}(X, G(Y))(*)$$

for all $X \in \mathscr{C}$, $Y \in \mathscr{D}$ that are functorial in X and Y.

Now let R be a ring, $\mathscr{C} = \mathscr{D}$ the category of R-modules.[1] Let $F \dashv G$ be an adjoint pair of functors between $\mathscr{C}$ and $\mathscr{D}$.

[1] One could just as well take the categories of modules over two possibly different rings.

1. Use the universal property of the (co-)product of modules to show that

$$F(M \oplus N) \cong F(M) \oplus F(N), \qquad G(M \times N) \cong G(M) \times G(N)$$

   for all R-modules M, N.

2. (It follows from Part 1 that the isomorphisms (*) are group homomorphisms. Try to find an argument for this; but you do not have to write this down.)

3. Prove that F is right exact and that G is left exact.

Hint for Part 3. You may use the characterization of left/right exactness in [AM] Proposition 2.9 (= [Alg2] Satz 3.14[2]).

References

[AM] M. Atiyah, I. Macdonald, *Introduction to Commutative Algebra*, Addison-Wesley.

[Alg2] U. Görtz, *Kommutative Algebra*, Vorlesungsskript[3], SS 2023.

[2] `https://math.ug/a2-ss23/sec-exakte-sequenzen.html#exaktmithomtesten`
[3] `https://math.ug/a2-ss23/`