

PhD Seminar: Shimura Varieties

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1 Introduction

1.1 Origins

Shimura varieties lie at the intersection of algebraic geometry, Hodge theory, Lie theory, and arithmetic geometry. They originate from the theory of elliptic modular functions and complex multiplication. The modern theory grew from the work of Shimura, Taniyama, and Weil on complex multiplication in the 1950s and Shimura's subsequent construction of canonical models. Deligne's work in 1971 and 1979 recast the theory in terms of reductive groups and Hodge structures; see [Mil05, Introduction]. The Baily–Borel theorem (1966) established the algebraicity and projective compactifiability of the arithmetic quotients underlying the theory.

The first example is the modular curve: a complex elliptic curve is $E_\tau = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$, and changing the lattice basis leads to $\mathrm{SL}_2(\mathbb{Z}) \backslash \mathfrak{H}$. In higher dimension, moduli spaces of principally polarized abelian varieties are described by Siegel space and its arithmetic quotient:

$$\mathfrak{H}_g = \{Z \in M_g(\mathbb{C}) : Z^t = Z, \operatorname{Im} Z > 0\}, \quad \mathcal{A}_g(\mathbb{C}) = \mathrm{Sp}_{2g}(\mathbb{Z}) \backslash \mathfrak{H}_g.$$

1.2 Hodge theory

For a complex abelian variety A , the integral lattice $H^1(A, \mathbb{Z})$ carries the Hodge decomposition

$$H^1(A, \mathbb{C}) = H^{1,0}(A) \oplus H^{0,1}(A), \quad H^{0,1}(A) = \overline{H^{1,0}(A)}.$$

A polarization supplies an integral alternating form with a positivity condition. Once the lattice and this form are fixed, varying the Hodge subspace $H^{1,0}$ gives precisely the period space $\mathfrak{H}_g$. Thus the moduli problem of abelian varieties can be studied through polarized Hodge structures.

More generally, a rational Hodge structure is encoded by a representation of the Deligne torus $S = \operatorname{Res}_{\mathbb{C}/\mathbb{R}} G_m$, acting on $V^{p,q}$ as $z^{-p}\bar{z}^{-q}$. This translates Hodge decompositions and tensors into algebraic-group language. Fixing Hodge numbers and a polarization gives a period domain, with Siegel space as the weight-one example. In smooth projective families, cohomology forms a local system, while the Hodge filtration varies holomorphically and satisfies Griffiths transversality. Locally, the resulting period maps take values in period domains. Shimura domains arise when the relevant group orbits are Hermitian symmetric and every tangent direction is horizontal: the corresponding adjoint Hodge types lie in $\{(-1, 1), (0, 0), (1, -1)\}$. This explains the Hodge-theoretic content of the Shimura axioms.

1.3 Shimura varieties

A Shimura datum (G, X) consists of a connected reductive group $G/\mathbb{Q}$ and a $G(\mathbb{R})$ -conjugacy class of homomorphisms $h : S \rightarrow G_{\mathbb{R}}$ satisfying the Hodge-type, Cartan, and nontriviality

conditions. Each connected component X^+ is a Hermitian symmetric domain. For a compact open subgroup $K \subset G(\mathbb{A}_f)$, the associated complex space is

$$\mathrm{Sh}_K(G, X)(\mathbb{C}) = G(\mathbb{Q}) \backslash (X \times G(\mathbb{A}_f) / K).$$

Here $\mathbb{A}_f$ denotes the finite adeles, and K records the finite level structure. The Siegel datum is $(\mathrm{GSp}_{2g}, \mathfrak{H}_g^\pm)$.

Structure theorem. Let $G(\mathbb{Q})_+$ denote the stabilizer of X^+ in $G(\mathbb{Q})$. There is a finite decomposition

$$\mathrm{Sh}_K(G, X)(\mathbb{C}) \simeq \coprod_{[a] \in G(\mathbb{Q})_+ \backslash G(\mathbb{A}_f) / K} \Gamma_a \backslash X^+, \quad \Gamma_a = G(\mathbb{Q})_+ \cap aKa^{-1}.$$

At neat level the quotients are smooth, and Baily–Borel equips them with quasi-projective algebraic structures and normal projective compactifications. A real representation $\rho : G_{\mathbb{R}} \rightarrow \mathrm{GL}(V_{\mathbb{R}})$ assigns to each $h \in X^+$ the Hodge structure defined by $\rho \circ h$. The Shimura axioms ensure that these Hodge filtrations vary holomorphically and satisfy Griffiths transversality. Thus a Shimura domain is a homogeneous parameter space whose tautological families are already variations of Hodge structure.

1.4 Goal of the seminar

The main goal is to understand the construction of Shimura varieties and to learn some of the central ideas in Hodge theory. Talks 1–5 introduce modular curves, algebraic groups, Hodge structures, period domains, and Hermitian symmetric domains. Talks 6–7 develop Shimura varieties. Talk 8 brings together the moduli and Shimura descriptions of Siegel modular varieties. The last four talks explore Mumford–Tate groups and Baily–Borel/Toroidal compactifications.

2 Program of the seminar

Each talk lasts 90 minutes, including discussion. Some outlines contain more material than can be covered in the allotted time. Speakers can adjust the scope and omit supplementary material or proofs as needed. Talks 1–8 form the core of the seminar; suggestions for alternative topics for the final four talks are welcome. Please feel free to discuss your talk with me beforehand, whether about the choice of material or specific mathematical details.

Talk 0: Introduction [14/10/26]

Give an overview of the seminar and the individual talks.

Talk 1: Modular curves [21/10/26]

Main references: [Ara26, §§5.1 and 5.3] and [Mil17b, §2].

Start with the lattice description $E_\tau = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$ of complex elliptic curves. Derive the $\mathrm{SL}_2(\mathbb{Z})$ -action on $\mathfrak{H}$ and the criterion for two such curves to be isomorphic. Describe $\mathfrak{H} = \mathrm{SL}_2(\mathbb{R})/\mathrm{SO}(2)$ and a standard fundamental domain for the $\mathrm{SL}_2(\mathbb{Z})$ -action. Introduce level structures and the principal congruence subgroup $\Gamma(N)$, and explain the moduli interpretation of $Y(N) = \Gamma(N) \backslash \mathfrak{H}$; explain why level $N \geq 3$ removes automorphisms.

Talk 2: Algebraic groups [28/10/26]

Main references: [Gao25, §1.2.1, §2.1] and [Mil17a].

This talk is mainly to introduce some notions in the theory of algebraic groups. Following [Mil17a, §§1a, 2a, 4a, 10b, 10d], define linear algebraic groups, their representations, and study their properties. Introduce the Lie algebra and derive the adjoint representation from conjugation. Give some examples of matrix groups.

Following [Gao25, §1.2.1] (See also [Edi14, §4.10]), define characters and cocharacters of tori and explain the weight decomposition for split tori. Then introduce the unipotent radical, reductive and semisimple groups, the centre, the derived group, and the adjoint quotient [Gao25, §2.1]. State the characterizations of reductive groups over $\mathbb{C}$ in [Gao25, Proposition 2.1.6, Corollary 2.1.7] (See also [Edi14, §§4.11–4.12]).

Talk 3: Hodge structures and the Deligne torus [04/11/26]

Main references: [Mil05, §2, pp. 24–28] and [Gao25, §§1.1–1.2].

Introduce Hodge decompositions, filtrations, morphisms, tensor operations, Tate twists, and polarizations. Explain [Mil05, Examples 2.4–2.5]. Prove the equivalence of definitions of Hodge structures defined by decomposition and by representations of the Deligne torus $S = \text{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m$ [Mil05, §2, equations (20)–(23)] [Gao25, §§1.2.2–1.2.3]. Describe the weight homomorphism, Weil operator and Hodge tensors. Mention [Mil05, Examples 2.8–2.10 and 2.12]. Show the equivalence of polarizable integral Hodge structures of type $(1, 0) + (0, 1)$ with complex Abelian varieties [Gao25, Examples 1.1.4 & 1.1.8].

Talk 4: Variations of Hodge structures and period domains [11/11/26]

Main reference: [Cat14, §§7.1–7.4].

Introduce local systems, flat vector bundles, and their correspondence [Cat14, §7.1]. For a smooth projective family, define the Kodaira–Spencer map and illustrate its local description [Cat14, §7.2.1]. Define a polarized variation of Hodge structure [Cat14, Definition 7.3.4], and explain its geometric origin, i.e., construct the Gauss–Manin connection and Hodge filtration as in [Cat14, §7.3.1].

Define period domains and their compact duals, and sketch their homogeneous descriptions [Cat14, Theorem 7.4.3]. Define period maps and explain their holomorphicity and Griffiths transversality [Cat14, Theorem 7.4.5], relating their differential to the Kodaira–Spencer map.

Talk 5: Hermitian symmetric domains [18/11/26]

Main references: [Mil05, §1 and §2, Theorem 2.14] and [Ker14, §1].

Define Hermitian symmetric spaces of noncompact type and explain their realization as bounded symmetric domains, following [Mil05, §1 pp. 5–10]. Use the upper half-plane as a toy example. Introduce Cartan involutions and explain their relation to polarizations [Mil05, Proposition 1.20]. Describe the circle homomorphism $U_1 = \{z \in \mathbb{C} : |z| = 1\} \rightarrow G$, and state the characterization in [Mil05, Theorem 1.21], outlining the proof. Translate its character condition into the adjoint Hodge types $(-1, 1), (0, 0), (1, -1)$. Explain the horizontality criterion [Gao25, Proposition 1.3.5] and why a connected classifying space with a horizontal tautological family is a Hermitian symmetric domain [Mil05, Theorem 2.14]. See also [Ker14, §1.B] for the Hodge-theoretic viewpoint.

Talk 6: Shimura data and their examples [25/11/26]

Main references: [Mil05, §§4–5] and [Gao25, §3.1].

Introduce congruence subgroups, the ring of finite adeles and compact open subgroups of $G(\mathbb{A}_f)$ [Mil05, §4, pp. 42–44]. Use [Mil05, §5] and [Gao25, §3.1] to define a Shimura datum (G, X) and interpret its axioms through Talk 5. Explain the complex structure on each component of X and the families of Hodge structures supplied by representations [Mil05, Corollary 5.8, Proposition 5.9]. Discuss zero-dimensional Shimura data and Shimura curves [Gao25, Examples 3.1.2, 3.1.4]; use [Mil05, Proposition 5.22, Example 5.24] for their basic properties. Following [Gao25, §3.1.2], define the Shimura variety

$$\mathrm{Sh}_K(G, X)(\mathbb{C}) = G(\mathbb{Q}) \backslash (X \times G(\mathbb{A}_f) / K)$$

for a compact open subgroup $K \subset G(\mathbb{A}_f)$, postponing the finite decomposition and algebraicity to Talk 7. Introduce morphisms of Shimura data and Hecke correspondences [Gao25, §3.1.2].

Talk 7: Shimura varieties [02/12/26]

Main references: [Mil05, §5 pp. 56–64] and [Gao25, §3.2].

First state the connectedness and approximation results needed below [Mil05, Theorems 5.2 & 5.4, Corollary 5.3]. Prove the finite decomposition of a Shimura variety into arithmetic quotients [Mil05, Lemmas 5.11–5.13]; see also [Gao25, Theorem 3.2.1].

Introduce arithmetic subgroups and neatness, and explain when a discrete group gives a smooth quotient of a Hermitian symmetric domain [Mil05, §3, Proposition 3.1]. State the Baily–Borel theorem [Mil05, Theorem 3.12] and deduce the algebraicity of Shimura varieties. Explain the role of Borel’s algebraicity theorem in making the maps from Talk 6 algebraic [Mil05, Theorems 3.14, 5.16]. Briefly discuss the additional Shimura axioms and the polarization criterion [Gao25, §3.1.1 and §2.2.2]. If time permits, state the description of connected components via the abelian quotient when G^{der} is simply connected [Mil05, Theorem 5.17].

Talk 8: Abelian varieties and Siegel modular varieties [09/12/26]

Main references: [Mil05, §6] and [Ara26, §4, §6].

Following [Mil05, §6, pp. 71–73] [Ara26, §§4.1–4.2], recall complex abelian varieties and polarizations. A principal polarization is a unimodular Riemann form on $H_1(A, \mathbb{Z})$. Choose a symplectic marking and derive the normalized period conditions $Z^t = Z$ and $\mathrm{Im} Z > 0$:

$$A_Z = \mathbb{C}^g / (\mathbb{Z}^g + Z\mathbb{Z}^g), \quad \mathfrak{H}_g \simeq \mathrm{Sp}_{2g}(\mathbb{R}) / U(g).$$

Derive the symplectic action and identify $\mathcal{A}_g(\mathbb{C}) = \mathrm{Sp}_{2g}(\mathbb{Z}) \backslash \mathfrak{H}_g$ as the coarse moduli space [Ara26, Lemmas 6.1.1–6.1.3, Theorem 6.2.1]. Explain level- N structures and the fine moduli space $\mathcal{A}_g(N) = \Gamma_g(N) \backslash \mathfrak{H}_g$ for $N \geq 3$ [Ara26, Theorem 6.2.4]. Construct the Siegel datum $(\mathrm{GSp}_{2g}, \mathfrak{H}_g^{\pm})$ and verify its axioms [Gao25, Example 3.1.3]. Finally, state the adelic moduli description of Siegel modular varieties [Gao25, Theorem 3.3.1].

Talk 9: Mumford–Tate groups [16/12/26]

Main references: [Del82, §3] and [Moo04, §§4–5].

Recall the representation-theoretic preliminaries [Del82, Proposition 3.1]. For a rational Hodge structure (V, h) , define $\mathrm{MT}(V) \subset \mathrm{GL}(V)$ as the smallest $\mathbb{Q}$ -algebraic subgroup through which h factors over $\mathbb{R}$, following [Moo04, §4.1]. Show the equivalent definition as the intersection of all stabilizers of Hodge classes [Del82, Proposition 3.4]. Briefly recall the theory of Tannakian categories. Introduce the tensor category generated by V and its forgetful fibre functor, and explain $\mathrm{MT}(V)$ as its Tannakian group [Moo04, Corollary 4.5]. Show that when the Hodge structure is polarizable, its Mumford–Tate group is reductive [Moo04, Proposition 4.9] [Del82, Proposition 3.6]. Compute some examples [Moo04, §§5.1–5.3].

Talk 10: Mumford–Tate groups in families [13/01/27]

Main reference: [Mil13, §6].

Given a holomorphic family of Hodge structures on a complex manifold S , show that the associated family of Mumford–Tate groups stays constant outside of a countable union of proper analytic subvarieties of S [Mil13, Proposition 6.14]. This is called the generic Mumford–Tate group. Introduce the notion of fixed-part property for a family of Hodge structures, and the algebraic monodromy group [Mil13, Definition 6.17]. State the relation between the generic Mumford–Tate group and algebraic monodromy group [Mil13, Theorem 6.19]. Mention the case of VHS from algebraic families [Mil13, Theorems 6.24–6.25]. Define period subdomains and discuss their basic properties, following [Mil13, §7, pp. 35–37]. Connect them to special subvarieties of a Shimura variety [Gao25, Definition 3.1.10, §3.2.6] and explain [Gao25, Theorems 3.4.1 & 3.4.2].

Supplementary resource for Talks 9–10: Soldatenkov’s lectures [Sol26].

Talk 11: Rational boundaries and the Baily–Borel compactification [20/01/27]

Main reference: [Gao25, §4, §6].

Explain the boundary components, compactification and Satake topology in the case of modular curves [Gao25, §4.1]. Introduce analytic boundary components in the general case [Gao25, §4.6], using Siegel space as the main higher-dimensional example. Outline the Baily–Borel construction following [Gao25, §6]: adjoin rational boundary components, equip the quotient with the Satake topology, and state the existence of its normal complex analytic structure. Explain how Poincaré–Eisenstein series provide enough automorphic forms to obtain a projective algebraic compactification. Relate this construction to the projective spectrum of the ring of automorphic forms [Mil05, Remark 3.13(c)].

Talk 12: Toroidal compactifications [27/01/27]

Main reference: [Gao25, §§7.1–7.2].

Following [Gao25, §7.1], give a brief introduction to toric geometry. Explain $G_m \subset \mathbb{A}^1$ and construct $\mathbb{P}^2$ from its fan. For a rational analytic boundary component, describe the fibration structure and the torus associated to it [Gao25, §7.2.1]. Explain how to define a rational polyhedral cone decomposition which is compatible with the action of an arithmetic subgroup [Gao25, §7.2.2]. Finally, introduce a Γ -admissible polyhedral decomposition and state the main theorem [Gao25, Theorem 7.2.3].

Supplementary resources for Talks 11–12: [Gor05, §§2–3] for the geometric ideas, [Lan, §4.2] for comparisons between compactifications, and the lectures [CZ23, Lan14].

References

- [Ara26] D. Arapura, *Complex Algebraic Geometry, Abelian Varieties, and Modular Varieties*. Course notes, 1 May 2026. <https://www.math.purdue.edu/~arapura/Class/abelCAG.pdf>.
- [Cat14] E. Cattani, Introduction to variations of Hodge structure. In E. Cattani, F. El Zein, P. A. Griffiths, and Lê Dũng Tráng (eds.), *Hodge Theory*, Mathematical Notes 49, Princeton University Press, Princeton, NJ, 2014, Chapter 7, pp. 297–332. <https://doi.org/10.23943/princeton/9780691161341.003.0007>.
- [CZ23] M. Corato Zanarella, *An Introduction to Compactifications of Shimura Varieties, with Examples*. Graduate Student Seminar, MSRI/SLMath, April 2023. Lecture video. <https://vimeo.com/819362099>.
- [Del82] P. Deligne (notes by J. S. Milne), Hodge cycles on abelian varieties. In P. Deligne, J. S. Milne, A. Ogus, and K.-y. Shih, *Hodge Cycles, Motives, and Shimura Varieties*, Lecture Notes in Mathematics 900, Springer-Verlag, Berlin, 1982, pp. 9–100. Revised online version, 1 October 2018. <https://www.jmilne.org/math/Documents/Deligne82.pdf>.
- [Edi14] B. Edixhoven, *Shimura Varieties: Notes for Some Introductory Lectures*. Lecture notes, 25 March 2014. https://websites.math.leidenuniv.nl/edixhoven/talks/2012/2012_06_shimura.pdf.
- [Gao25] Z. Gao, *An Introduction to Shimura Varieties and Their Compactifications*. Lecture notes, 8 December 2025. <https://ziyangjeremygao.github.io/LectureNotes/ShimuraVarieties.pdf>.
- [Gor05] M. Goresky, Compactifications and cohomology of modular varieties. In J. Arthur, D. Ellwood, and R. Kottwitz (eds.), *Harmonic Analysis, the Trace Formula, and Shimura Varieties*, Clay Mathematics Proceedings 4, American Mathematical Society, Providence, RI, 2005, pp. 551–582. <https://www.math.ias.edu/~goresky/pdf/compactifications.pdf>.
- [Ker14] M. Kerr, Shimura varieties: a Hodge-theoretic perspective. In E. Cattani, F. El Zein, P. A. Griffiths, and Lê Dũng Tráng (eds.), *Hodge Theory*, Mathematical Notes 49, Princeton University Press, Princeton, NJ, 2014, pp. 525–566. <https://www.math.wustl.edu/~matkerr/SV.pdf>.
- [Lan] K.-W. Lan, *An Example-Based Introduction to Shimura Varieties*. Online manuscript, to appear in the proceedings of the ETHZ Summer School on Motives and Complex Multiplication; accessed 26 September 2026. <https://www.kwlan.org/articles/intro-sh-ex.pdf>.
- [Lan14] K.-W. Lan, *An Example-Based Introduction to Shimura Varieties and Their Compactifications*. Three lectures, Introductory Workshop: New Geometric Methods in Number Theory and Automorphic Forms, MSRI, 18–20 August 2014. Lecture videos: [Lecture 1](#), [Lecture 2](#), [Lecture 3](#).
- [Mil05] J. S. Milne, Introduction to Shimura varieties. In J. Arthur, D. Ellwood, and R. Kottwitz (eds.), *Harmonic Analysis, the Trace Formula, and Shimura Varieties*, Clay Mathematics Proceedings 4, American Mathematical Society, Providence, RI, 2005, pp. 265–378. Revised online version, 16 September 2017. <https://www.jmilne.org/math/xnotes/svi.pdf>.
- [Mil13] J. S. Milne, Shimura varieties and moduli. In G. Farkas and I. Morrison (eds.), *Handbook of Moduli*, Vol. II, Advanced Lectures in Mathematics 25, International Press, Somerville, MA, and Higher Education Press, Beijing, 2013, pp. 467–548. References in this program use the numbering of the online manuscript, version 2.00, 30 April 2011. <https://www.jmilne.org/math/xnotes/svh.pdf>.

- [Mil17a] J. S. Milne, *Algebraic Groups: The Theory of Group Schemes of Finite Type over a Field*. Cambridge Studies in Advanced Mathematics 170, Cambridge University Press, Cambridge, 2017.
<https://www.jmilne.org/math/Books/iAG2017.pdf>.
- [Mil17b] J. S. Milne, *Modular Functions and Modular Forms*. Course notes, version 1.31, 22 March 2017.
<https://www.jmilne.org/math/CourseNotes/MF.pdf>.
- [Moo04] B. Moonen, *An Introduction to Mumford–Tate Groups*. Lecture notes, 11 May 2004.
<https://www.math.ru.nl/~bmoonen/Lecturenotes/MTGps.pdf>.
- [Sol26] A. Soldatenkov, *Mumford–Tate Groups in Hodge Theory*. Minicourse, IMPA Summer School, 2026. Lecture videos.
<https://www.youtube.com/playlist?list=PLo4jXE-LdDTSzcTajnmQcjswxspaMCCL->.